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How to Use Advanced Algebra II

This module contains a table of every module within the three books of Kenny Felder's course on "Algebra II", with links to the modules.

Over a period of time, I have developed a set of in-class assignments, homeworks, and lesson plans, that work for me and for other people who have tried them. The complete set comprises three separate books that work together:

- The [Homework and Activities Book](#) contains in-class and homework assignments that are given to the students day-by-day.
- The [Concepts Book](#) provides conceptual explanations, and is intended as a reference or review guide for students; it is not used when teaching the class.
- The [Teacher's Guide](#) provides lesson plans; it is your guide to how I envisioned these materials being used when I created them (and how I use them myself).

Instructors should note that this book probably contains more information than you will be able to cover in a single school year. I myself do not teach from every chapter in my own classes, but have chosen to include these additional materials to assist you in meeting your own needs. As you will likely need to cut some sections from the book, I strongly recommend that you spend time early on to determine which modules are most important for your state requirements and personal teaching style.

One more warning is important: these materials were designed for an **Advanced** Algebra II course. For such a course, I hope this will provide you with ready-to-use textbook and lesson plans. If you are teaching a Standard or Remedial-level course, these materials will still be useful, but you will probably have to cut or reduce some of the most conceptual material, and supplement it with more drill-and-practice than I provide.

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Function Concepts

This module provides an introduction to the concept of functions.

The unit on functions is the most important in the Algebra II course, because it provides a crucial transition point. Roughly speaking...

- Before Algebra I, math is about **numbers**.
- Starting in Algebra I, and continuing into Algebra II, math is about **variables**.
- Beginning with Algebra II, and continuing into Calculus, math is about **functions**.

Each step builds on the previous step. Each step expands the ability of mathematics to model behavior and solve problems. And, perhaps most crucially, each step can be frightening to a student. It can be very intimidating for a beginning Algebra student to see an entire page of mathematics that is covered with letters, with almost no numbers to be found!

Unfortunately, many students end up with a very vague idea of what variables are (“That’s when you use letters in math”) and an even more vague understanding of functions (“Those things that look like $f(x)$ or something”). If you leave yourself with this kind of vague understanding of the core concepts, the lessons will make less and less sense as you go on: you will be left with the feeling that “I just can’t do this stuff” without realizing that the problem was all the way back in the idea of a variable or function.

The good news is, variables and functions both have very specific meanings that are not difficult to understand.

What is a Variable?

This module defines what a variable is and what it does.

A variable is a letter that stands for a number you don't know, or a number that can change.

A few examples:

Example:

Good Examples of Variable Definitions

- “Let p be the number of people in a classroom.”
- “Let A be John's age, measured in years.”
- “Let h be the number of hours that Susan has been working.”

In each case, the letter stands for a very specific number. However, we use a letter instead of a number because we don't know the specific number. In the first example above, different classrooms will have different numbers of people (so p can be different numbers in different classes); in the second example, John's age is a specific and well-defined number, but we don't know what it is (at least not yet); and in the third example, h will actually change its value every hour. In all three cases, we have a good reason for using a letter: it represents a number, but we cannot use a specific number such as “ -3 ” or “ $4\frac{1}{2}$ ”.

Example:

Bad Examples of Variable Definitions

- “Let n be the nickels.”
- “Let M be the number of minutes in an hour.”

The first error is by far the most common. Remember that a variable always stands for a number. “The nickels” are not a number. Better definitions would be: “Let n be the number of nickels” or “Let n be the total value of the nickels, measured in cents” or “Let n be the total mass of the nickels, measured in grams.”

The second example is better, because “number of minutes in an hour” is a number. But there is no reason to call it “The Mysterious Mr. M” because we already know what it is. Why use a letter when you just mean “60”?

Bad variable definitions are one of the most common reasons that students get stuck on word problems—or get the wrong answer. The first type of error illustrated above leads to variable confusion: n will end up being used for “number of nickels” in one equation and “total value of the nickels” in another, and you end up with the wrong answer. The second type of error is more harmless—it won’t lead to wrong answers—but it won’t help either. It usually indicates that the student is asking the wrong question (“What can I assign a variable to?”) instead of the right question (“What numbers do I need to know?”)

Variables aren’t all called x . Get used to it.

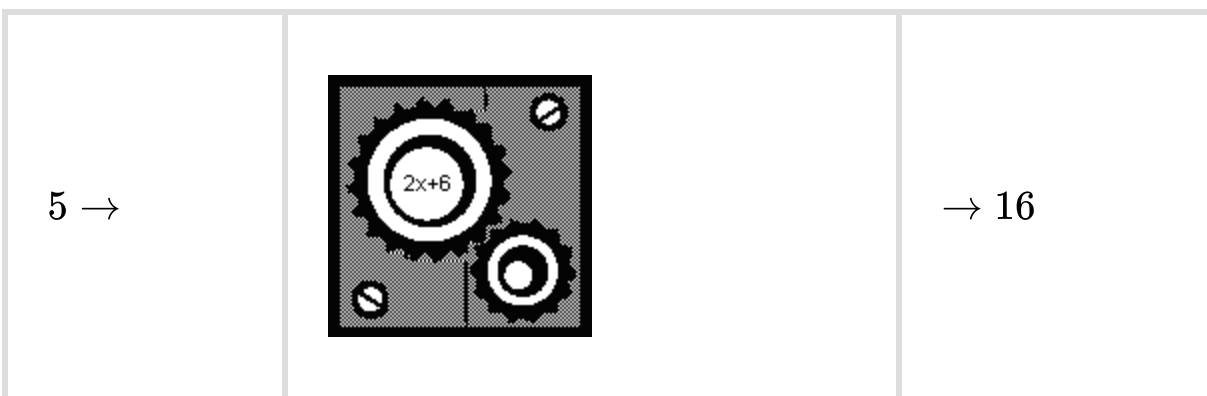
Many students expect all variables to be named x , with possibly an occasional guest appearance by y . In fact, variables can be named with practically any letter. Uppercase letters, lowercase letters, and even Greek letters are commonly used for variable names. Hence, a problem might start with “Let H be the home team’s score and V be the visiting team’s score.”

If you attempt to call both of these variables x , it just won’t work. You could in principle call one of them x and the other y , but that would make it more difficult to remember which variable goes with which team. It is important to become comfortable using a wide range of letters. (I do, however, recommend avoiding the letter o whenever possible, since it looks like the number 0.)

Function Concepts -- What is a Function?

A function is neither a number nor a variable: it is a **process for turning one number into another**. For instance, “Double and then add 6” is a function. If you put a 4 into that function, it comes out with a 14. If you put a $\frac{1}{2}$ into that function, it comes out with a 7.

The traditional image of a function is a machine, with a slot on one side where numbers go in and a slot on the other side where numbers come out.



A number goes in. A number comes out. The function is the machine, the process that turns 4 into 14 or 5 into 16 or 100 into 206.

The point of this image is that the function is not the numbers, but the machine itself—the **process**, not the results of the process.

The primary purpose of “The Function Game” that you play on Day 1 is to get across this idea of a numerical process. In this game, one student (the “leader”) is placed in the role of a function. “Whenever someone gives you a number, you double that number, add 6, and give back the result.” It should be very clear, as you perform this role, that you are **not** modeling a number, a variable, or even a list of numbers. You are instead modeling a process—or an algorithm, or a recipe—for turning numbers into other numbers. That is what a function is.

The function game also contains some more esoteric functions: “Respond with -3 no matter what number you are given,” or “Give back the lowest prime number that is greater than or equal to the number you were given.” Students playing the function game often ask “Can a function **do** that?” The answer is always yes (with one caveat mentioned below). So another purpose of the function game is to expand your idea of what a function can do. Any process that consistently turns numbers into other numbers, is a function.

By the way—having defined the word “function” I just want to say something about the word “equation.” An “equation” is when you “equate” two things—that is to say, set them equal. So $x^2 - 3$ is a function, but it is not an equation. $x^2 - 3 = 6$ is an equation. An “equation” always has an equal sign in it.

The Rule of Consistency

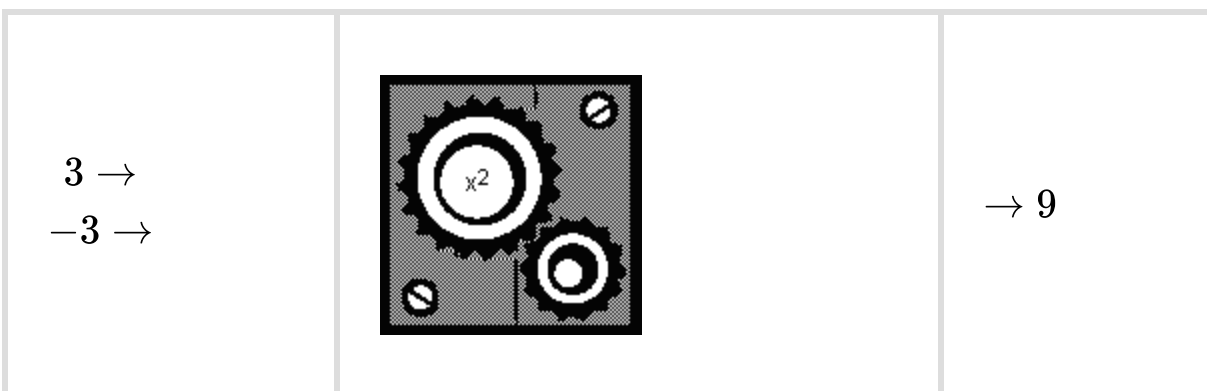
This module discusses how functions must be consistent.

There is only one limitation on what a function can do: **a function must be consistent.**

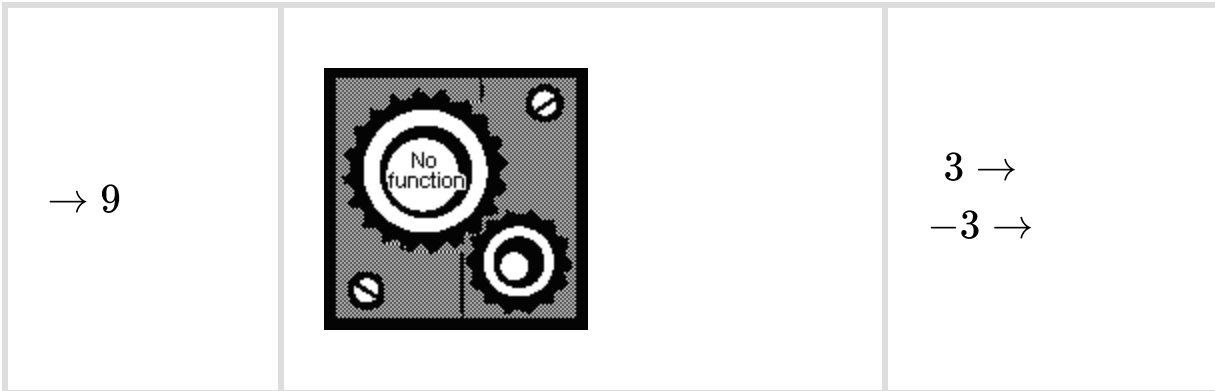
For instance, the function in the above drawing is given a 5, and gives back a 16. That means this particular function turns 5 into 16—always. That particular function can never take in a 5 and give back a 14. This “rule of consistency” is a very important constraint on the nature of functions.

Note: This rule does not treat the inputs and outputs the same!

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9 (two different inputs, same output). That is allowed. However, it is not reversible! If you take a 9 and turn it into both a 3 and a -3 (two different outputs, same input), you are not a function.



If 3 goes in, 9 comes out. If -3 goes in, 9 also comes out. No problem: x^2 is a function.



If 9 goes in, both -3 and 3 come out. This **violates the rule of consistency**: no function can do this

This asymmetry has the potential to cause a great deal of confusion, but it is a very important aspect of functions.

Four Ways to Represent a Function

This module discusses how it is possible to describe functions in four different ways: graphically, verbally, algebraically, and numerically.

Modern Calculus texts emphasize that a function can be expressed in four different ways.

1. **Verbal** - This is the first way functions are presented in the function game: “Double and add six.”
2. **Algebraic** - This is the most common, most concise, and most powerful representation: $2x + 6$. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).
3. **Numerical** - This can be done as a list of value pairs, as $(4,14)$ — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as (x,y) points used in graphing.)
4. **Graphical** - This is discussed in detail in the section on graphing.

These are **not** four different types of functions: they are four different views of the same function. One of the most important skills in Algebra is converting a function between these different forms, and this theme will recur in different forms throughout the text.

Domain and Range

This module defines the domain and range of a function.

Consider the function $y = \sqrt{x}$. If this function is given a 9 it hands back a 3. If this function is given a 2 it hands back...well, it hands back $\sqrt{2}$, which is approximately 1.4. The answer cannot be specified exactly as a fraction or decimal, but it is a perfectly good answer nonetheless.

On the other hand, what if this function is handed -4 ? There is no $\sqrt{-4}$, so the function has no number to hand back. If our function is a computer or calculator, it responds with an error message. So we see that this function is able to respond to the numbers 9 and 2, but it is not able to respond in any way to the number -4 . Mathematically, we express this by saying that 9 and 2 are in the “domain” of the square root function, and -4 is **not** in the domain of this function.

Domain

The **domain** of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

A square root cannot successfully act on a negative number. We say that “The domain of \sqrt{x} is all numbers
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such that
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” meaning that if you give this function zero or a positive number, it can act on it; if you give this function a negative number, it cannot.

A subtler example is the function $y = \sqrt{x + 7}$. Does this function have the same domain as the previous function? No, it does not. If you hand this function a -4 it successfully hands back $\sqrt{3}$ (about 1.7). -4 is in the domain of this function. On the other hand, if you hand this function a -8 it attempts to take $\sqrt{-1}$ and fails; -8 is **not** in the domain of this function. If you play with a few more numbers, you should be able to convince yourself that the domain of this function is all numbers x such that $x \geq -7$.

You are probably familiar with two mathematical operations that are **not allowed**. The first is, you are not allowed to take the square root of a negative number. As we have seen, this leads to restrictions on the domain of any function that includes square roots.

The second restriction is, you are not allowed to divide by zero. This can also restrict the domain of functions. For instance, the function $y = \frac{1}{x^2-4}$ has as its domain all numbers **except** $x = 2$ and $x = -2$. These two numbers both cause the function to attempt to divide by 0, and hence fail. If you ask a calculator to plug $x = 2$ into this function, you will get an error message.

So: if you are given a function, how can you find its domain? Look for any number that puts a negative number under the square root; these numbers are **not** in the domain. Look for any number that causes the function to divide by zero; these numbers are not in the domain. All other numbers **are** in the domain.

Function	Domain	Comments
\sqrt{x}	$x \geq 0$	You can take the square root of 0, or of any positive number, but you cannot take the square root of a negative number.

Function	Domain	Comments
$\sqrt{x + 7}$	$x \geq -7$	If you plug in any number greater than or equal to -7 , you will be taking a legal square root. If you plug in a number less than -7 , you will be taking the square root of a negative number. This domain can also be understood graphically: the graph $y = \sqrt{x}$ has been moved 7 units to the left. See “horizontal permutations” below.
$\frac{1}{x}$	$x \neq 0$	In other words, the domain is “all numbers except 0.” You are not allowed to divide by 0. You are allowed to divide by anything else.
$\frac{1}{x-3}$	$x \neq 3$	If $x = 3$ then you are dividing by 0, which is not allowed. If $x = 0$ you are dividing by -3 , which is allowed. So be careful! The rule is not “when you are dividing, x cannot be 0.” The rule is “ x can never be any value that would put a 0 in the denominator.”
$\frac{1}{x^2-4}$	$x \neq \pm 2$	Or, “ x can be any number except 2 or -2 .” Either of these x values will put a 0 in the denominator, so neither one is allowed.

Function	Domain	Comments
$2^x + x^2 - 3x + 4$	All numbers	You can plug any x value into this function and it will come back with a number.
$\frac{\sqrt{x-3}}{x-5}$	$x \geq 3$ $x \neq 5$	In words, the domain is all numbers greater than or equal to 3, except the number 5 . Numbers less than 3 put negative numbers under the square root; 5 causes a division by 0.

You can confirm all these results with your calculator; try plugging numbers into these functions, and see when you get errors!

A related concept is **range**.

Range

The **range** of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come **out** of the function.

To illustrate this example, let us return to the function $y = \sqrt{x + 7}$. Recall that we said the domain of this function was all numbers x such that $x \geq -7$; in other words, you are allowed to put any number greater than or equal to -7 into this function.

What numbers might come out of this function? If you put in a -7 you get out a 0. ($\sqrt{0} = 0$) If you put in a -6 you get out $\sqrt{1} = 1$. As you increase the x value, the y values also increase. However, if you put in $x = -8$ nothing comes out at all. Hence, the range of this function is all numbers y such that $y \geq 0$. That is, this function is capable of handing back 0 or any positive number, but it will never hand back a negative number.

It's easy to get the words domain and range confused—and it's important to keep them distinct, because although they are **related** concepts, they are **different** from each other. One trick that sometimes helps is to remember that, in everyday useage, “your domain” is your home, your land—it is where you begin. A function begins in its own domain. It ends up somewhere out on the range.

A different notation for domain and range

Domains and ranges above are sometimes expressed as intervals, using the following rules:

- Parentheses () mean “an interval starting or ending here, but not including this number”
- Square brackets [] mean “an interval starting or ending here, including this number”

This is easiest to explain with examples.

This notation...	...means this...	...or in other words
$(-3,5)$	All numbers between -3 and 5 , not including -3 and 5 .	$-3 < x < 5$
$[-3,5]$	All numbers between -3 and 5 , including -3 and 5 .	$-3 \leq x \leq 5$
$[-3,5)$	All numbers between -3 and 5 , including -3 but not 5 .	$-3 \leq x < 5$

This notation...	...means this...	...or in other words
$(-\infty, 10]$	All numbers less than or equal to 10.	$x \leq 10$
$(23, \infty)$	All numbers greater than 23.	$x > 23$
$(-\infty, 4)$ $(4, \infty)$	All numbers less than 4, and all numbers greater than 4. In other words, all numbers except 4 .	$x \neq 4$

Why are functions so important that they form the heart of math from Algebra II onward?

Functions are used whenever **one variable depends on another variable**. This relationship between two variables is the most important in mathematics. It is a way of saying “If you tell me what x is, I can tell you what y is.” We say that y “depends on” x , or y “is a function of” x .

A few examples:

Example:

Function Concepts -- Functions in the Real World

- "The area of a circle depends on its radius."
- "The amount of money Alice makes depends on the number of hours she works."
- "Max threw a ball. The height of the ball depends on how many seconds it has been in the air."

In each case, there are two variables. Given enough information about the scenario, you could assert that **if you tell me this variable, I will tell you that one**. For instance, suppose you know that Alice makes \$100 per day. Then we could make a chart like this.

If Alice works this many days...	...she makes this many dollars
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If Alice works this many days...	...she makes this many dollars
0	0
1	100
1½	150
8	800

If you tell me how long she has worked, I will tell you how much money she has made. Her earnings “depend on” how long she works.

The two variables are referred to as the **dependent variable** and the **independent variable**. The dependent variable is said to “depend on” or “be a function of” the independent variable. “The height of the ball is a function of the time.”

Example:

Bad Examples of Functional Relationships

- "The number of Trojan soldiers depends on the number of Greek soldiers."
- "The time depends on the height of the ball."

The first of these two examples is by far the most common. It is simply not true. There may be a relationship between these two quantities—for instance, the **sum** of these two variables might be the total number of soldiers, and the **difference** between these two quantities might suggest whether the battle will be a fair one. But there is no **dependency** relationship—that is, no way to say “If you tell me the number of Greek

soldiers, I will tell you the number of Trojan soldiers”—so this is not a function.

The second example is subtler: it confuses the **dependent** and the **independent** variables. The height depends on the time, not the other way around. More on this in the discussion of “Inverse Functions”.

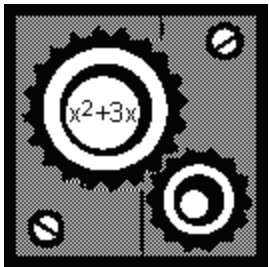
Function Notation

This module describes notation for functions.

Function Notation

Functions are represented in math by parentheses. When you write $f(x)$ you indicate that the variable f is a function of—or depends on—the variable x .

For instance, suppose $f(x) = x^2 + 3x$. This means that f is a function that takes whatever you give it, and squares it, and multiplies it by 3, and adds those two quantities.

<p>7 → 10 → x → y → a dog →</p>		<p>→ $f(7) = 7^2 + 3(7) = 70$ → $f(10) = 10^2 + 3(10) = 130$ → $f(x) = x^2 + 3x$ → $f(y) = y^2 + 3y$ → $f(\text{dog}) = (\text{dog})^2 + 3(\text{dog})$ (*not in the domain)</p>
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The notation $f(7)$ means “plug the number 7 into the function f .” It does **not** indicate that you are multiplying f times 7. To evaluate $f(7)$ you take the function $f(x)$ and replace all occurrences of the variable x with the number 7. If this function is given a 7 it will come out with a 70.

If we write $f(y) = y^2 + 3y$ we have **not specified a different function**. Remember, the function is not the variables or the numbers, it is the process. $f(y) = y^2 + 3y$ also means “whatever number comes in, square it, multiply it by 3, and add those two quantities.” So it is a different way of writing the same function.

Just as many students expect all variables to be named x , many students—and an unfortunate number of parents—expect all functions to be named f . The correct rule is that—whenever possible—**functions, like variables, should be named descriptively**. For instance, if Alice makes \$100/day, we might write:

- Let m equal the amount of money Alice has made (measured in dollars)
- Let t equal the amount of time Alice has worked (measured in days)
- Then, $m(t) = 100t$

This last equation should be read “ m is a function of t (or m depends on t). Given any value of the variable t , you can multiply it by 100 to find the corresponding value of the variable m .”

Of course, this is a very simple function! While simple examples are helpful to illustrate the concept, it is important to realize that very complicated functions are also used to model real world relationships. For instance, in Einstein’s Special Theory of Relativity, if an object is going very fast, its mass is multiplied by $\frac{1}{\sqrt{1 - \frac{v^2}{9 \cdot 10^{16}}}}$. While this can look extremely

intimidating, it is just another function. The speed v is the independent variable, and the mass m is dependent. Given any speed v you can determine how much the mass m is multiplied by.

Algebraic Generalizations

This module discusses the generalizations of algebraic functions and their implications.

When you have a “generalization,” you have one **broad fact** that allows you to assume many **specific** facts as examples.

Example:

Generalization: “Things fall down when you drop them.”

Specific facts, or examples:

- Leaves fall down when you drop them
- Bricks fall down when you drop them
- Tennis balls fall down when you drop them

If **any one** of the individual statements does not work, the generalization is invalid. (This generalization became problematic with the invention of the helium balloon.)

Scientists tend to work **empirically**, meaning they start with the specific facts and work their way back to the generalization. Generalizations are valued in science because they bring order to apparently disconnected facts, and that order in turn suggests underlying theories.

Mathematicians also spend a great deal of time looking for generalizations. When you have an “algebraic generalization” you have one **algebraic** fact that allows you to assume many **numerical** facts as examples.

Consider, for instance, the first two functions in the function game.

1. Double the number, then add six.
2. Add three to the number, then double.

These are very different “recipes.” However, their inclusion in the function game is a bit unfair, because—here comes the generalization—**these two functions will always give the same answer**. Whether the input is positive or negative, integer or fraction, small or large, these two functions will mimic each other perfectly. We can express this generalization in words.

Example:


Generalization: If you plug a number into the function **double and add six**, and plug the same number into the function **add three and double**, the two operations will give the same answer.


Specific facts, or examples:

- If you double -5 and add six; or, if you add -5 to 3 and then double; you end up with the same answer.
- If you double 13 and add six; or, if you add 13 to 3 and then double; you end up with the same answer.

There is literally an infinite number of specific claims that fit this pattern. We don't need to prove or test each of these claims individually: once we have proven the generalization, we know that **all** these facts must be true.

We can express this same generalization pictorially by showing two “function machines” that always do the same thing.

$-5 \rightarrow$ $0 \rightarrow$ $13 \rightarrow$	 <p>A diagram of a function machine. It is a square box with a gear on the left and a smaller circle on the right. Above the box is the expression $2x+6$. The input x enters from the left, goes through the gear, and then through the smaller circle before exiting on the right.</p>	$\rightarrow 2(-5) + 6 = -4$ $\rightarrow 2(0) + 6 = 6$ $\rightarrow 2(13) + 6 = 32$
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$-5 \rightarrow$ $0 \rightarrow$ $13 \rightarrow$	 <p>A diagram of a function machine. It is a square box with a gear on the left and a smaller circle on the right. Above the box is the expression $2(x+3)$. The input x enters from the left, goes through the gear, then through the smaller circle, and finally through a larger gear before exiting on the right.</p>	$\rightarrow 2(-5) + 6 = -4$ $\rightarrow 2(0) + 6 = 6$ $\rightarrow 2(13) + 6 = 32$
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But the most common way to express this generalization is algebraically, by asserting that these two functions **equal** each other.

Equation:

$$2x + 6 = 2(x + 3)$$

Many beginning Algebra II students will recognize this as the distributive property. Given $2(x + 3)$ they can correctly turn it into $2x + 6$. **But they often fail to realize what this equality means—that given the same input, the two functions will always yield the same output.**

Example:

Generalization: $2x + 6 = 2(x + 3)$

Specific facts, or examples:

- $(2 \times -5) + 6 = 2 \times (-5 + 3)$
- $(2 \times 0) + 6 = 2 \times (0 + 3)$
- $(2 \times 13) + 6 = 2 \times (13 + 3)$

It's worth stopping for a moment here to think about the $=$ symbol. Whenever it is used, $=$ indicates that two things are the same. However, the following two equations use the $=$ in very different ways.

Equation:

$$2x^2 + 5x = 3$$

Equation:

$$\frac{2x^2 - 18}{x + 3} = 2x - 6$$

In the first equation, the $=$ challenges you to **solve for x** . “Find all the x values that make this equation true.” The answers in this case are $x = \frac{1}{2}$ and $x = -3$. If you

plug in either of these two x -values, you get a true equation; for any other x -value, you get a false equation.

The second equation cannot be solved for x ; the $=$ sign in this case is asserting an equality that is true for **any** x -value. Let's try a few.

Example:		
Generalization: $\frac{2x^2-18}{x+3} = 2x - 6$		
Specific facts, or examples:		
$x = 3$	$\frac{2(3)^2-18}{(3)+3} = \frac{18-18}{6} = 0$	$2(3) - 6 = 0 \checkmark$
$x = -2$	$\frac{2(-2)^2-18}{(-2)+3} = \frac{8-18}{1} = -10$	$2(-2) - 6 = -10 \checkmark$
$x = 0$	$\frac{2(0)^2-18}{(0)+3} = \frac{0-18}{3} = -6$	$2(0) - 6 = -6 \checkmark$
$x = \frac{1}{2}$	$\frac{2(\frac{1}{2})^2-18}{(\frac{1}{2})+3} = \frac{\frac{1}{2}-18}{\frac{7}{2}} = \left(\frac{-35}{2}\right)\left(\frac{2}{7}\right) = -5$	$2\left(\frac{1}{2}\right) - 6 = -5 \checkmark$

With a calculator, you can attempt more difficult values such as $x = -26$ or $x = \pi$; in every case, the two formulas will give the same answer. When we assert that two very different functions will always produce the same answers, we are making a very powerful generalization.

Exception: $x = -3$ is outside the domain of one of these two functions. In this important sense, the two functions are not in fact equal. Take a moment to make sure you understand why this is true!

Such generalizations are very important because they allow us to **simplify**.

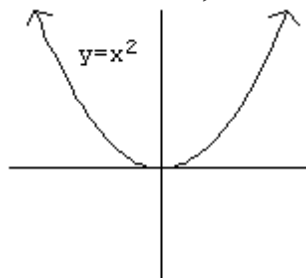
Suppose that you were told “I am going to give you a hundred numbers. For each number I give you, square it, then double the answer, then subtract eighteen, then divide by the original number plus three.” This kind of operation comes up all the time. But you would be quite relieved to discover that you can accomplish the same task by simply doubling each number and subtracting 6! The generalization in this case is $\frac{2x^2-18}{x+3} = 2x - 6$; you will be creating exactly this sort of generalization in the chapter on Rational Expressions.

Graphing

This module describes how to graph basic functions.

Graphing, like algebraic generalizations, is a difficult topic because many students know **how to do it** but are not sure **what it means**.

For instance, consider the following graph:



If I asked you “Draw the graph of $y = x^2$ ” you would probably remember how to plot points and draw the shape.

But suppose I asked you this instead: “Here’s a function, $y = x^2$. And here’s a shape, that sort of looks like a U. What do they actually have to do with each other?” This is a harder question! What does it **mean** to graph a function?

The answer is simple, but it has important implications for a proper understanding of functions. Recall that every point on the plane is designated by a unique (x,y) pair of coordinates: for instance, one point is $(5,3)$. We say that its x -value is 5 and its y -value is 3.

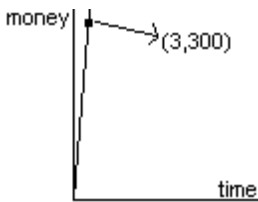
A few of these points have the particular property that their y -values are the square of their x -values. For instance, the points $(0,0)$, $(3,9)$, and $(-5,25)$ all have that property. $(5,3)$ and $(-2, -4)$ do not.

The graph shown—the pseudo-U shape—is **all the points in the plane that have this property**. Any point whose y -value is the square of its x -value is on this shape; any point whose y -value is not the square of its x -value is not on this shape. Hence, glancing at this shape gives us a complete visual picture of the function $y = x^2$ **if we know how to interpret it correctly**.

Graphing Functions

Remember that every function specifies a relationship between two variables. When we graph a function, we put the **independent variable** on the x -axis, and the **dependent variable** on the y -axis.

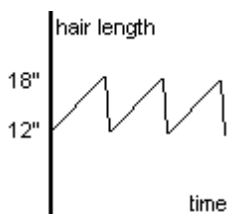
For instance, recall the function that describes Alice's money as a function of her hours worked. Since Alice makes \$12/hour, her financial function is $m(t) = 12t$. We can graph it like this.



This simple graph has a great deal to tell us about Alice's job, if we read it correctly.

- The graph contains the point (3,300). What does that tell us? That after Alice has worked for three hours, she has made \$300.
- The graph goes through the origin (the point (0,0)). What does that tell us? That when she works 0 hours, Alice makes no money.
- The graph exists only in the first quadrant. What does that tell us? On the mathematical level, it indicates the **domain** of the function ($t \geq 0$) and the **range** of the function ($m \geq 0$). In terms of the situation, it tells us that Alice cannot work negative hours or make negative money.
- The graph is a straight line. What does that tell us? That Alice makes the same amount of money every day: every day, her money goes up by \$100. (\$100/day is the **slope** of the line—more on this in the section on linear functions.)

Consider now the following, more complicated graph, which represents **Alice's hair length as a function of time** (where time is now measured in weeks instead of hours).



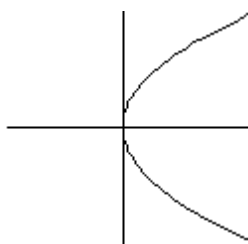
What does this graph $h(t)$ tell us? We can start with the same sort of simple analysis.

- The graph goes through the point $(0,12)$. This tells us that at time $(t = 0)$, Alice's hair is 12" long.
- The range of this graph appears to be $12 \leq h \leq 18$. Alice never allows her hair to be shorter than 12" or longer than 18".

But what about the shape of the graph? The graph shows a gradual incline up to 18", and then a precipitous drop back down to 12"; and this pattern repeats throughout the shown time. The most likely explanation is that Alice's hair grows slowly until it reaches 18", at which point she goes to the hair stylist and has it cut down, within a very short time (an hour or so), to 12". Then the gradual growth begins again.

The rule of consistency, graphically

Consider the following graph.



This is our earlier "U" shaped graph ($y = x^2$) turned on its side. This might seem like a small change. But ask this question: what is y when $x = 3$? This question has two answers. This graph contains the points $(3, -9)$ and $(3,9)$. So when $x = 3$, y is both 9 and -9 on this graph.

This violates the only restriction on functions—the rule of consistency. Remember that the x -axis is the independent variable, the y -axis the dependent. In this case, one “input” value (3) is leading to two different “output” values ($-9,9$) We can therefore conclude that this graph **does not represent a function at all.** No function, no matter how simple or complicated, could produce this graph.

This idea leads us to the “vertical line test,” the graphical analog of the rule of consistency.

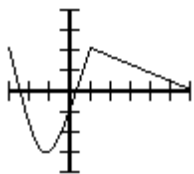
The Vertical Line Test

If you can draw **any** vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

It is important to understand that the vertical line test is not a new rule! It is the graphical version of the rule of consistency. If any vertical line touches a graph in two places, then the graph has two different y -values for the same x -value, and this is the only thing that functions are not allowed to do.

What happens to the graph, when you add 2 to a function?

Suppose the following is the graph of the function $y = f(x)$.



$$y = f(x);$$

Contains

the

following

points

(among

others):

$(-3,2),$
 $(-1, -3)$
 $, (1,2),$
 $(6,0)$

We can see from the graph that the domain of the graph is $-3 \leq x \leq 6$ and the range is $-3 \leq y \leq 2$.

Question: **What does the graph of $y = f(x) + 2$ look like?**

This might seem an impossible question, since we do not even know what the function $f(x)$ is. But we don't need to know that in order to plot a few points.

x	$f(x)$	$f(x + 2)$	so $y = f(x)$ contains this point	and $y = f(x) + 2$ contains this point
-3	2	4	$(-3,2)$	$(-3,4)$
-1	-3	-1	$(-1, -3)$	$(-1, -1)$
1	2	4	$(1,2)$	$(1,4)$
6	0	2	$(6,0)$	$(6,2)$

If you plot these points on a graph, the pattern should become clear. Each point on the graph is **moving up by two**. This comes as no surprise: since you added 2 to each y-value, and adding 2 to a y-value moves any point up by 2. So the new graph will look identical to the old, only moved up by 2.

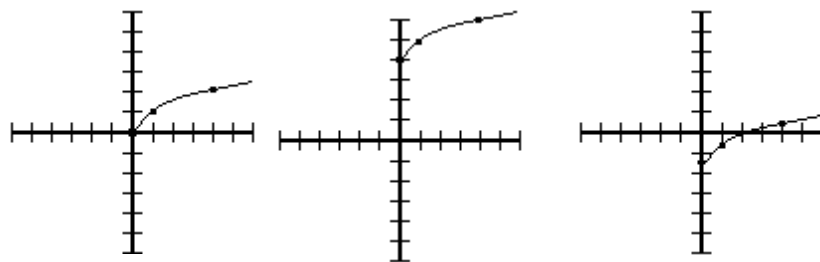
$$y = f(x) \quad y = f(x) + 2; \text{ All } y\text{-values are 2 higher}$$



In a similar way, it should be obvious that if you **subtract** 10 from a function, the graph moves **down** by 10. Note that, in either case, the **domain** of the function is the same, but the **range** has changed.

These permutations work for **any function**. Hence, given the graph of the function $y = \sqrt{x}$ below (which you could generate by plotting points), you can produce the other two graphs **without** plotting points, simply by moving the first graph up and down.

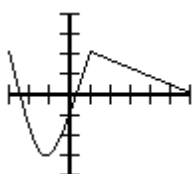
$$y = \sqrt{x} \quad y = \sqrt{x} + 4 \quad y = \sqrt{x} - 1\frac{1}{2}$$



Other vertical permutations

Adding or subtracting a constant from $f(x)$, as described above, is one example of a **vertical permutation**: it moves the graph up and down. There are other examples of vertical permutations.

For instance, what does **doubling** a function do to a graph? Let's return to our original function:



$$y = f(x)$$

What does the graph $y = 2f(x)$ look like? We can make a table similar to the one we made before.

x	$f(x)$	$2f(x)$	so $y = 2f(x)$ contains this point
-3	2	4	$(-3, 4)$
-1	-3	-6	$(-1, -6)$
1	2	4	$(1, 4)$
6	0	0	$(6, 0)$

In general, the high points move higher; the low points move lower. The entire graph is **vertically stretched**, with each point moving farther away from the x-axis.

$$y = f(x) \quad y = 2f(x); \text{ All } y\text{-values are doubled}$$



Similarly, $y = \frac{1}{2}f(x)$ yields a graph that is vertically compressed, with each point moving toward the x-axis.

Finally, what does $y = -f(x)$ look like? All the positive values become negative, and the negative values become positive. So, point by point, the entire graph flips over the x-axis.

$$y = f(x) \quad y = -f(x); \text{ All } y\text{-values change sign}$$

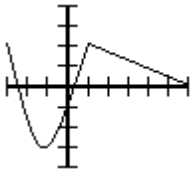


What happens to the graph, when you add 2 to the x value?

Vertical permutations affect the y-value; that is, the output, or the function itself. Horizontal permutations affect the x-value; that is, the numbers that

come in. They often do the opposite of what it naturally seems they should.

Let's return to our original function $y = f(x)$.



$$y = f(x);$$

Contains
the
following
points
(among
others):
 $(-3, 2)$,
 $(-1, -3)$
, $(1, 2)$,
 $(6, 0)$

Suppose you were asked to graph $y = f(x + 2)$. Note that this is **not the same** as $f(x) + 2$! The latter is an instruction to run the function, and then add 2 to all results. But $y = f(x + 2)$ is an instruction to add 2 to every x -value **before plugging it into the function**.

- $f(x) + 2$ changes y , and therefore shifts the graph vertically
- $f(x + 2)$ changes x , and therefore shifts the graph horizontally.

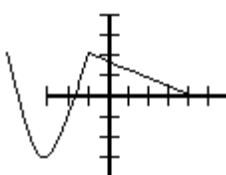
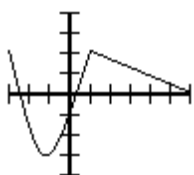
But which way? In analogy to the vertical permutations, you might expect that adding two would shift the graph to the right. But let's make a table of values again.

x	$x + 2$	$f(x + 2)$	so $y = f(x + 2)$ contains this point
$-\frac{5}{5}$	-3	$f(-3)=2$	$(-5,2)$
$-\frac{3}{3}$	-1	$f(-1)=-3$	$(-3, - 3)$
$-\frac{1}{1}$	1	$f(1)=2$	$(-1,2)$
4	6	$f(6)=0$	$(4,0)$

This is a very subtle, very important point—please follow it closely and carefully! First of all, make sure you understand where all the numbers in that table came from. Then look what happened to the original graph.

Note: The original graph $f(x)$ contains the point $(6,0)$; therefore, $f(x + 2)$ contains the point $(4,0)$. The point has moved two spaces to the left.

$y = f(x)$ $y = f(x+2)$; Each point is shifted to the left



You see what I mean when I say horizontal permutations “often do the opposite of what it naturally seems they should”? **Adding** two moves the graph to the **left**.

Why does it work that way? Here is my favorite way of thinking about it. $f(x - 2)$ is an instruction that says to each point, “look two spaces to your left, and copy what the original function is doing **there**.” At $x = 5$ it does what $f(x)$ does at $x = 3$. At $x = 10$, it copies $f(8)$. And so on. Because it is always copying $f(x)$ to its **left**, this graph ends up being a copy of $f(x)$ moved to the **right**. If you understand this way of looking at it, all the rest of the horizontal permutations will make sense.

Of course, as you might expect, subtraction has the opposite effect: $f(x - 6)$ takes the original graph and moves it 6 units to the **right**. In either case, these horizontal permutations affect the **domain** of the original function, but not its **range**.

Other horizontal permutations

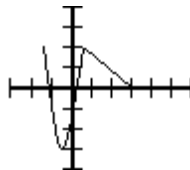
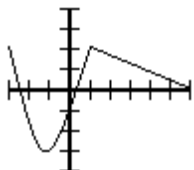
Recall that $y = 2f(x)$ **vertically stretches** a graph; $y = \frac{1}{2}f(x)$ **vertically compresses**. Just as with addition and subtraction, we will find that the horizontal equivalents work backward.

x	$2x$	$f(2x)$	so $y = 2f(x)$ contains this point
$-1\frac{1}{2}$	-3	2	$(-1\frac{1}{2}, 2)$
$-\frac{1}{2}$	-1	-3	$(-\frac{1}{2}; -3)$
$\frac{1}{2}$	1	2	$(\frac{1}{2}; 2)$
3	6	0	$(3, 0)$

The original graph $f(x)$ contains the point $(6, 0)$; therefore, $f(2x)$ contains the point $(3, 0)$. Similarly, $(-1; -3)$ becomes $(-\frac{1}{2}; -3)$. Each point is

closer to the y-axis; the graph has horizontally **compressed**.

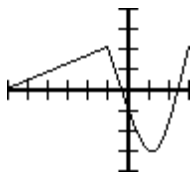
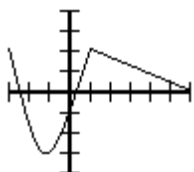
$y = f(x)$ $y = f(2x)$; Each point is twice as close to the y -axis



We can explain this the same way we explained $f(x - 2)$. In this case, $f(2x)$ is an instruction that says to each point, “Look outward, at the x -value that is double yours, and copy what the original function is doing **there**.” At $x = 5$ it does what $f(x)$ does at $x = 10$. At $x = -3$, it copies $f(-6)$. And so on. Because it is always copying $f(x)$ **outside** itself, this graph ends up being a copy of $f(x)$ moved **inward**; ie a compression. Similarly, $f(\frac{1}{2}x)$ causes each point to look inward toward the y -axis, so it winds up being a **horizontally stretched** version of the original.

Finally, $y = f(-x)$ does precisely what you would expect: it flips the graph around the y -axis. $f(-2)$ is the old $f(2)$ and vice-versa.

$y = f(x)$ $y = f(-x)$; Each point flips around the y -axis

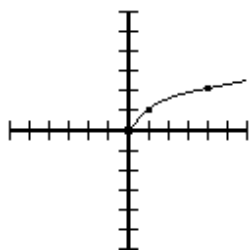


All of these permutations do not need to be memorized: only the general principles need to be understood. But once they are properly understood,

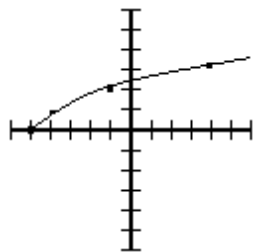
even a complex graph such as $y = -2(x + 3)^2 + 5$ can be easily graphed. You take the (known) graph of $y = x^2$, flip it over the x-axis (because of the negative sign), stretch it vertically (the 2), move it to the left by 3, and move it up 5.

With a good understanding of permutations, and a very simple list of known graphs, it becomes possible to graph a wide variety of important functions. To complete our look at permutations, let's return to the graph of $y = \sqrt{x}$ in a variety of flavors.

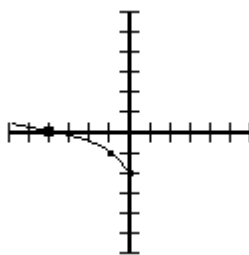
$y = \sqrt{x}$;
Generated by plotting points;
Contains (0,0), (1,1), (4,2);
Domain: $x \geq 0$;
Range: $y \geq 0$;
Range: $y \geq 0$



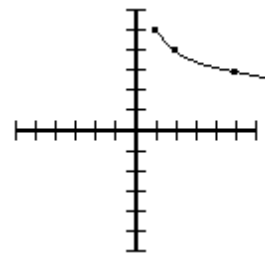
$y = \sqrt{x + 5}$;
Shifted 5 units to the left;
Contains (-5,0), (-4,1), (-1,2);
Domain: $x \geq -5$;
Range: $y \geq 0$



$y = \sqrt{-x} - 2$;
Flipped horizontally, shifted down 2;
Contains (0, -2), (-1, -1), (-4,0);
Domain: $x \leq 0$;
Range: $y \geq -2$



$y = -\sqrt{x - 1} + 5$;
Flipped vertically, shifted 1 to the right and 5 up;
Contains (1,5), (2,4), (5,3);
Domain: $x \geq 1$;
Range: $y \leq 5$



Lines

This module discusses lines and their uses, and slope.

Most students entering Algebra II are already familiar with the basic mechanics of graphing lines. Recapping very briefly: the equation for a line is $y = mx + b$ where b is the y -intercept (the place where the line crosses the y -axis) and m is the slope. If a linear equation is given in another form (for instance, $4x + 2y = 5$), the easiest way to graph it is to rewrite it in $y = mx + b$ form (in this case, $y = -2x + 2\frac{1}{2}$).

There are two purposes of reintroducing this material in Algebra II. The first is to frame the discussion as **linear functions modeling behavior**. The second is to deepen your understanding of the important concept of slope.

Consider the following examples. Sam is a salesman—he earns a commission for each sale. Alice is a technical support representative—she earns \$100 each day. The chart below shows their bank accounts over the week.

After this many days (t)	Sam's bank account (S)	Alice's bank account (A)
0 (*what they started with)	\$75	\$750
1	\$275	\$850
2	\$375	\$950
3	\$450	\$1,050
4	\$480	\$1,150

After this many days (t)	Sam's bank account (S)	Alice's bank account (A)
5	\$530	\$1,250

Sam has some extremely good days (such as the first day, when he made \$200) and some extremely bad days (such as the second day, when he made nothing). Alice makes exactly \$100 every day.

Let d be the number of days, S be the number of dollars Sam has made, and A be the number of dollars Alice has made. Both S and A are functions of time. But $s(t)$ is **not a linear function**, and $A(t)$ is **a linear function**.

Linear Function

A function is said to be “linear” if **every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount**.

Once you know that Alice's bank account function is linear, there are only two things you need to know before you can predict her bank account on any given day.

- How much money she started with (\$750 in this example). This is called the **y -intercept**.
- How much she makes each day (\$100 in this example). This is called the **slope**.

y -intercept is relatively easy to understand. Verbally, it is where the function starts; graphically, it is where the line crosses the y -axis.

But what about slope? One of the best ways to understand the idea of slope is to convince yourself that all of the following definitions of slope are actually the same.

Definitions of Slope		
In our example	In general	On a graph
Each day, Alice's bank account increases by 100. So the slope is 100.	Each time the independent variable increases by 1, the dependent variable increases by the slope.	Each time you move to the right by 1, the graph goes up by the slope.
Between days 2 and 5, Alice earns \$300 in 3 days. $300/3=100$. Between days 1 and 3, she earns \$200 in 2 days. $200/2=100$.	Take any two points. The change in the dependent variable, divided by the change in the independent variable, is the slope.	Take any two points. The change in y divided by the change in x is the slope. This is often written as $\frac{\Delta y}{\Delta x}$, or as $\frac{\text{rise}}{\text{run}}$
The higher the slope, the faster Alice is making money.	The higher the slope, the faster the dependent variable increases.	The higher the slope, the faster the graph rises as you move to the right.

So slope does not tell you where a graph is, but how quickly it is rising. Looking at a graph, you can get an approximate feeling for its slope without any numbers. Examples are given below.

A slope of 1: each time you go **over** 1,

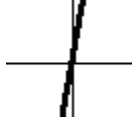
A steep slope of perhaps 3 or 4

A gentle slope of perhaps $\frac{1}{2}$.

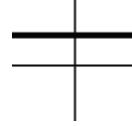
A horizontal line has a slope of 0: each time you go

This goes **down** as you move left to right. So the slope is negative. It is

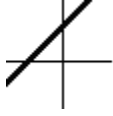
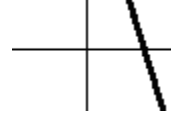
you also
go **up** 1



over 1, you
don't go **up**
at all!



steep: maybe
a -2 .



Function Concepts -- Composite Functions

You are working in the school cafeteria, making peanut butter sandwiches for today’s lunch.


- The more classes the school has, the more children there are.
- The more children there are, the more sandwiches you have to make.
- The more sandwiches you have to make, the more pounds (lbs) of peanut butter you will use.
- The more peanut butter you use, the more money you need to budget for peanut butter.

...and so on. Each sentence in this little story is a function. Mathematically, if c is the number of classes and h is the number of children, then the first sentence asserts the existence of a function $h(c)$.

The principal walks up to you at the beginning of the year and says “We’re considering expanding the school. If we expand to 70 classes, how much money do we need to budget? What if we expand to 75? How about 80?” For each of these numbers, you have to calculate each number from the previous one, until you find the final budget number.

# Classes		# Children		# Sandwiches		lb.		\$\$
-----------	--	------------	--	--------------	--	-----	--	------

But going through this process each time is tedious. What you want is one function that puts the entire chain together: “You tell me the number of classes, and I will tell you the budget.”

# Classes		\$\$
-----------	---	------

This is a **composite function**—a function that represents in one function, the results of an entire **chain of dependent functions**. Since such chains are very common in real life,

finding composite functions is a very important skill.

How do you make a composite Function?

We can consider how to build composite functions into the function game that we played on the first day. Suppose Susan takes any number you give her, quadruples it, and adds 6. Al takes any number you give him and divides it by 2. Mathematically, we can represent the two functions like this:

Equation:

$$S(x) = 4x + 6$$

Equation:

$$A(x) = \frac{x}{2}$$

To create a chain like the one above, we give a number to Susan; she acts on it, and gives the resulting number to Al; and he then acts on it and hands back a third number.

$$3 \rightarrow \text{Susan} \rightarrow S(3) = 18 \rightarrow \text{Al} \rightarrow A(18) = 9$$

In this example, we are plugging $S(3)$ —in other words, 18—into Al's function. In general, for any x that comes in, we are plugging $S(x)$ **into** $A(x)$. So we could represent the entire process as $A(S(x))$. This notation for composite functions is really nothing new: it means that you are plugging $S(x)$ into the A function.

But in this case, recall that $S(x) = 4x + 6$. So we can write:

Equation:

$$A(S(x)) = \frac{S(x)}{2} = \frac{4x + 6}{2} = 2x + 3$$

What happened? We've just discovered a shortcut for the entire process. When you perform the operation $A(S(x))$ —that is, when you perform the Al function on the result of the Susan function—you are, in effect, doubling and adding 3. For instance, we saw earlier that when we started with a 3, we ended with a 9. Our composite function does this in one step:

$$3 \rightarrow 2x + 3 \rightarrow 9$$

Understanding the meaning of composite functions requires real thought. It requires understanding the idea that this variable depends on that variable, which in turn depends on the other variable; and how that idea is translated into mathematics. **Finding** composite functions, on the other hand, is a purely mechanical process—it requires practice, but no

creativity. Whenever you are asked for $f(g(x))$, just plug the $g(x)$ function into the $f(x)$ function and then simplify.

Example:

Building and Testing a Composite Function

$$f(x) = x^2 - 4x$$

$$g(x) = x + 2$$

What is $f(g(x))$?

- To find the composite, plug $g(x)$ into $f(x)$, just as you would with any number.

$$f(g(x)) = (x + 2)^2 - 4(x + 2)$$

- Then simplify.

$$f(g(x)) = (x^2 + 4x + 4) - (4x + 8)$$

$$f(g(x)) = x^2 - 4$$

- Let's test it. $f(g(x))$ means do g , then f . What happens if we start with $x = 9$?

$$7 \rightarrow g(x) \rightarrow 7 + 2 = 9 \rightarrow f(x) \rightarrow (9)^2 - 4(9) = 45$$

- So, if it worked, our **composite function** should do all of that in one step.

$$7 \rightarrow x^2 - 4 = (7)^2 - 4 = 45 \checkmark \text{ It worked!}$$

There is a different notation that is sometimes used for composite functions. This book will consistently use $f(g(x))$ which very naturally conveys the idea of “plugging $g(x)$ into $f(x)$.” However, you will sometimes see the same thing written as $f \circ g(x)$, which more naturally conveys the idea of “doing one function, and then the other, in sequence.” The two notations mean the same thing.

Inverse Functions

This module describes what inverse functions are and how they can be used.

Let's go back to Alice, who makes \$100/day. We know how to answer questions such as "After 3 days, how much money has she made?" We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: "If Alice has made \$300, how many hours has she worked?" This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

- **If a function answers the question:** "Alice worked this long, how much money has she made?" **then its inverse answers the question:** "Alice made this much money, how long did she work?"
- **If a function answers the question:** "I have this many spoons, how much do they weigh?" **then its inverse answers the question:** "My spoons weigh this much, how many do I have?"
- **If a function answers the question:** "How many hours of music fit on 12 CDs?" **then its inverse answers the question:** "How many CDs do you need for 3 hours of music?"

How do you recognize an inverse function?

Let's look at the two functions above:

Equation:

$$m(t) = 100t$$

Equation:

$$t(m) = m/100$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

$$3 \rightarrow m(t) = 100t \rightarrow 300$$

$$300 \rightarrow t(m) = m/100 \rightarrow 3$$

✓ **Inverse functions**

Of course, this makes logical sense. The first line above says that “If Alice works 3 hours, she makes \$300.” The second line says “If Alice made \$300, she worked 3 hours.” It’s the same statement, made in two different ways.

But this “reversal” property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:

Equation:

$$f(x) = 3x + 7$$

Equation:

$$g(x) = \frac{1}{3}x - 7$$

They **look** like inverses, don’t they? But let’s test and find out.

$$2 \rightarrow 3x + 7 \rightarrow 13$$

$$13 \rightarrow \frac{3}{x} - 7 \rightarrow \frac{13}{3} - 7 \rightarrow -\frac{8}{3}$$

✗ **Not inverse functions**

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:

Equation:

$$f(x) = 3x + 7$$

Equation:

$$g(x) = \frac{1}{3}(x - 7)$$

Let's run our test of inverses on these two functions.

$2 \rightarrow 3x + 7 \rightarrow 13$
$13 \rightarrow \frac{1}{3}(x - 7) \rightarrow 2$
✓ Inverse functions

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:

Equation:

$$F(C) = \left(\frac{9}{5}\right)C + 32$$

Equation:

$$C(F) = \left(\frac{5}{9}\right)(F - 32)$$

where C is the Celsius temperature and F the Fahrenheit. If you plug $100^\circ C$ into the first equation, you find that it is $212^\circ F$. If you ask the second equation about $212^\circ F$, it of course converts that back into $100^\circ C$.

The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f(f^{-1}(x)) = x$.

Inverse Function

$f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b , then $f^{-1}(x)$ must turn b into a . More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

Finding an inverse function

In examples above, we saw that if $f(x) = 3x + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y . So to find the inverse function, **switch the x and y values, and then solve for y**.

Example:

Building and Testing an Inverse Function

1. Find the inverse function of $f(x) = \frac{2x-3}{5}$
 - a. Write the function as $y = \frac{2x-3}{5}$
 - b. Switch the x and y variables. $x = \frac{2y-3}{5}$
 - c. Solve for y . $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
2. Test to make sure this solution fills the definition of an inverse function.
 - a. Pick a number, and plug it into the original function.
 $9 \rightarrow f(x) \rightarrow 3$.
 - b. See if the inverse function reverses this process.
 $3 \rightarrow f^{-1}(x) \rightarrow 9$. ✓ It worked!

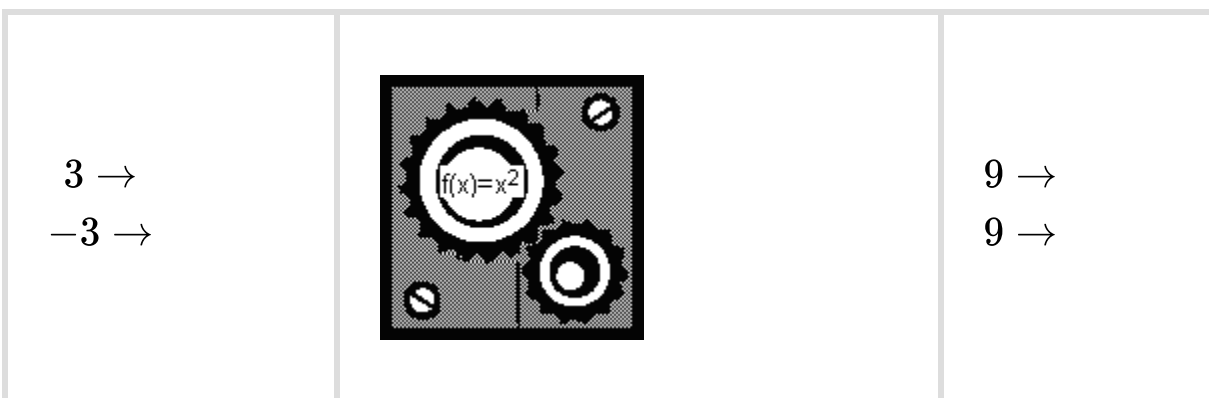
Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

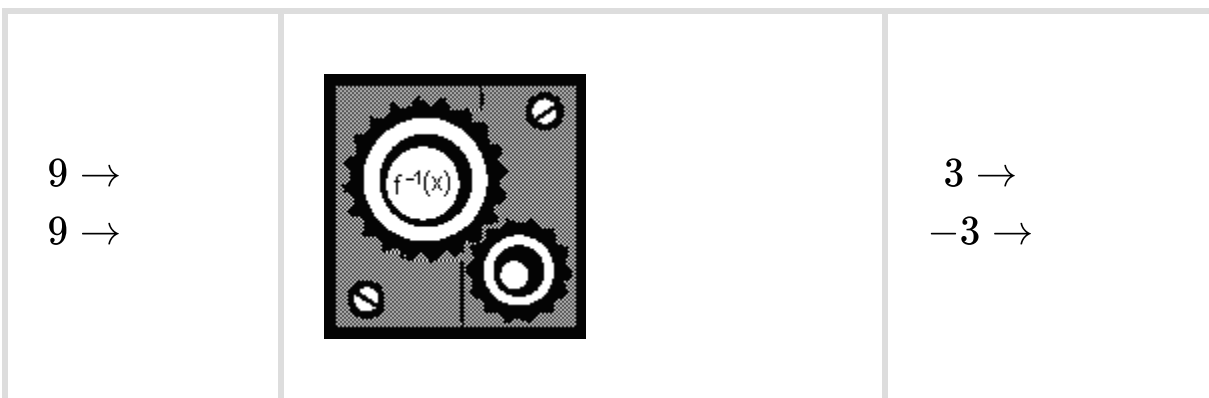
Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and -3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!



If 3 goes in, 9 comes out. If -3 goes in, 9 also comes out. No problem:



But its inverse would have to turn 9 into both 3 and -3. No function can do

this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask “How much money did Alice make in 3 days?” we must surely be able to ask “How long did it take Alice to make \$500?” When would you have a function that cannot be inverted?

Let’s go back to this example:

Recall the example that was used earlier: “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.” The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as “After 3 seconds, where is the ball?”

The inverse question would be “At what time was the ball 10 feet in the air?” The problem with that question is, it may well have **two answers!**

The ball is here...	...after this much time has elapsed
10 ft	2 seconds (*on the way up)
10 ft	5 seconds (*on the way back down)

So what does that mean? Does it mean we can’t ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a

“function” would not be consistent: one question would produce multiple answers.

Inequalities

This module introduces the concept of inequalities.

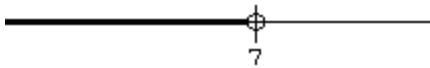
The symbols for inequalities are familiar:

- $x < 7$ “ x is less than 7”
- $x > 7$ “ x is greater than 7”
- $x \leq 7$ “ x is less than or equal to 7”
- $x \geq 7$ “ x is greater than or equal to 7”

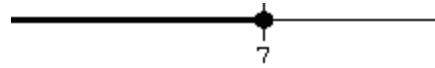
If you have trouble remembering which is which, it may be helpful to remember that the larger side of the $<$ symbol always goes with the larger number. Hence, when you write $x < 7$ you can see that the 7 is the larger of the two numbers. Some people think of the $<$ symbol as an alligator’s mouth, which always opens toward the largest available meal!

Visually, we can represent these inequalities on a number line. An open circle \circ is used to indicate a boundary that is not a part of the set; a closed circle \bullet is used for a boundary that is a part of the set.

Includes all numbers less than 7, but not 7; $x < 7$; $(-\infty, 7)$



Includes all numbers less than 7, **and** 7 itself; $x \leq 7$; $(-\infty, 7]$



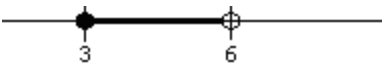
AND and OR

More complicated intervals can be represented by combining these symbols with the logical operators **AND** and **OR**.

For instance, “ $x \geq 3$ **AND** $x < 6$ ” indicates that x must be **both** greater-than-or-equal-to 3, **and** less-than 6. A number only belongs in this set if it meets **both** conditions. Let’s try a few numbers and see if they fit.

Sample number	$x \geq 3$	$x < 6$	$x \geq 3$ AND $x < 6$ (both true)
$x = 8$	Yes	No	No
$x = 0$	No	Yes	No
$x = 4$	Yes	Yes	Yes

We can see that a number must be **between** 3 and 6 in order to meet this AND condition.



$$x \geq 3 \text{ AND } x < 6$$

All numbers that are greater-than-or-equal-to 3, **and** are also less than 6; $3 \leq x < 6$

This type of set is sometimes represented concisely as $3 \leq x < 6$, which visually communicates the idea that x is **between** 3 and 6. This notation always indicates an AND relationship.

“ $x < 3$ OR $x \geq 6$ ” is the exact opposite. It indicates that x must be **either** less-than 3, **or** greater-than-or-equal-to 6. Meeting both conditions is OK, but it is not necessary.

Sample number	$x < 3$	$x \geq 6$	$x < 3$ OR $x \geq 6$ (either one or both true)
$x = 8$	No	Yes	Yes
$x = 0$	Yes	No	Yes
$x = 4$	No	No	No

Visually, we can represent this set as follows:



All numbers
that are either
less than 3, or
greater-than-or-
equal-to 6;
 $x < 3$ OR $x \geq 6$

Both of the above examples are meaningful ways to represent useful sets. It is possible to put together many combinations that are perfectly logical, but are not meaningful or useful. See if you can figure out simpler ways to write each of the following conditions.

1. $x \geq 3$ AND $x > 6$
2. $x \geq 3$ OR $x > 6$
3. $x < 3$ AND $x > 6$
4. $x > 3$ OR $x < 6$

If you are not sure what these mean, try making tables of numbers like the ones I made above. Try a number below 3, a number between 3 and 6, and a

number above 6. See when each condition is true. You should be able to convince yourself of the following:

1. The first condition above is filled by any number greater than 6; it is just a big complicated way of writing $x > 6$.
2. Similarly, the second condition is the same as $x \geq 3$.
3. The third condition is **never** true.
4. The fourth condition is **always** true.

I have to pause here for a brief philosophical digression. The biggest difference between a good math student, and a poor or average math student, is that the good math student works to **understand** things; the poor student tries to **memorize rules** that will lead to the right answer, without actually understanding them.

The reason this unit (Inequalities and Absolute Values) is right here at the beginning of the book is because it distinguishes sharply between these two kinds of students. Students who try to understand things will follow the previous discussion of AND and OR and will think about it until it makes sense. When approaching a new problem, they will try to make logical sense of the problem and its solution set.

But many students will attempt to learn a set of mechanical rules for solving inequalities. These students will often end up producing nonsensical answers such as the four listed above. Instead of thinking about what their answers mean, they will move forward, comfortable because “it looks sort of like the problem the teacher did on the board.”

If you have been accustomed to looking for mechanical rules to follow, now is the time to begin changing your whole approach to math. It’s not too late!!! Re-read the previous section carefully, line by line, and make sure each sentence makes sense. Then, as you work problems, think them through in the same way: **not** “whenever I see this kind of problem the answer is an **and**” but instead “What does AND mean? What does OR mean? Which one correctly describes this problem?”

All that being said, there are still a few hard-and-fast rules that I will point out as I go. These rules are useful—but they do not relieve you of the

burden of thinking.

One special kind of OR is the symbol \pm . Just as \geq means “greater than OR equal to,” \pm means “plus OR minus.” Hence, if $x^2 = 9$, we might say that $x = \pm 3$; that is, x can be either 3, or -3 .

Another classic sign of “blind rule-following” is using this symbol with inequalities. What does it mean to say $x < \pm 3$? If it means anything at all, it must mean “ $x < 3$ OR $x < -3$ ”; which, as we have already seen, is just a sloppy shorthand for $x < 3$. If you find yourself using an inequality with a \pm sign, go back to think again about the problem.

Note: Inequalities and the \pm symbol don’t mix.

Solving Inequalities

Inequalities are solved just like equations, with one key exception.

Note: Whenever you **multiply or divide by a negative number**, the sign changes.

You can see how this rule affects the solution of a typical inequality problem:

$3x + 4 > 5x + 10$	An “inequality” problem
--------------------	-------------------------

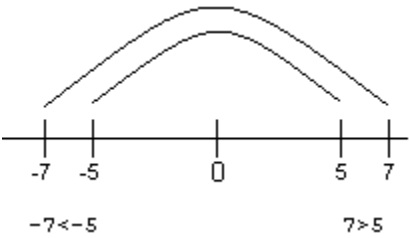
$-2x + 4 > 10$	subtract $5x$ from both sides
$-2x > 6$	subtract 4 from both sides
$x < -3$	divide both sides by -2, and change sign!

As always, being able to solve the problem is important, but even more important is knowing **what the solution means**. In this case, we have concluded that any number less than -3 will satisfy the original equation, $3x + 4 > 5x + 10$. Let's test that.

$x = -4$:	$3(-4) + 4 > 5(-4) + 10$	$-8 > -10$	Yes.
$x = -2$:	$3(-2) + 4 > 5(-2) + 10$	$-2 > 0$	No.

As expected, $x = -4$ (which is less than -3) works; $x = -2$ (which is not) does not work.

Why do you reverse the inequality when multiplying or dividing by a negative number? Because negative numbers are backward! 5 is greater than 3 , but -5 is **less than** -3 . Multiplying or dividing by negative numbers moves you to the other side of the number line, where everything is backward.



Multiplying by -1
moves you "over
the rainbow" to the
land where
everything is
backward!

Absolute Value Equations

This module introduces absolute value equations.

Absolute value is one of the simplest functions—and paradoxically, one of the most problematic.

On the face of it, nothing could be simpler: it just means “whatever comes in, a positive number comes out.”

Equation:

$$| 5 | = 5$$

Equation:

$$| -5 | = 5$$

Absolute values seem to give us permission to ignore the whole nasty world of negative numbers and return to the second grade when all numbers were positive.

But consider these three equations. They look very similar—only the number changes—but the solutions are completely different.

$ x = 10$	$ x = -10$	$ x = 0$
$x = 10$ works.	$x = 10$ doesn't work.	$x = 0$ is the only solution.
Hey, so does $x = -10$!	Neither does $x = -10$.	

Concisely,
 $x = \pm 10$.

Hey...**absolute values are never negative!**

Three Simple Absolute Value Equations

We see that the first problem has **two solutions**, the second problem has **no solutions**, and the third problem has **one solution**. This gives you an example of how things can get confusing with absolute values—and how you can solve things if you **think** more easily than with **memorized rules**.

For more complicated problems, follow a three-step approach.

1. Do the algebra to isolate the absolute value.
2. Then, **think it through** like the simpler problems above.
3. Finally, do more algebra to isolate x .

In my experience, most problems with this type of equation do not occur in the first and third step. And they do not occur because students try to think it through (second step) and don't think it through correctly. They occur because students try to take "shortcuts" to avoid the second step entirely.

Example:

Absolute Value Equation (No Variable on the Other Side)

$$3|2x + 1| - 7 = 5$$

- **Step 1:** Algebraically isolate the absolute value
 - $3|2x + 1| = 12$
 - $|2x + 1| = 4$
- **Step 2:** Think!
 - For the moment, forget about the quantity $2x + 1$; just think of it as **something**. The absolute value of "something" is 4. So, in analogy to what we did before, the "something" can either be 4, or -4 . So that gives us two possibilities...

- $2x + 1 = 4$
- $2x + 1 = -4$

• **Step 3:** Algebraically solve (both equations) for x

- $2x = 3$ or $2x = -5$
- $x = \frac{3}{2}$ or $x = -\frac{5}{2}$

So this problem has two answers: $x = \frac{3}{2}$ and $x = -\frac{5}{2}$

Example:

Absolute Value Equation (No Variable on the Other Side)

$$\frac{6|x-2|}{5} + 7 = 4$$

- **Step 1:** Algebraically isolate the absolute value
Equation:

$$\frac{6|x-2|}{5} = -3$$

Equation:

$$6|x-2| = -15$$

Equation:

$$|x-2| = \frac{-15}{6} = \frac{-5}{2}$$

- **Step 2: Think!** The absolute value of “something” is $-2\frac{1}{2}$. But wait —absolute values are **never** negative! It can’t happen! So we don’t even need a third step in this case: the equation is impossible.
- **No solution.**

The “think” step in the above examples was relatively straightforward, because there were no variables on the right side of the equation. When there are variables on the right side, you temporarily “pretend” that the right side of the equation is a positive number, and break the equation up accordingly. However, there is a price to be paid for this slight of hand: you have to check your answers, because **they may not work even if you do your math correctly.**

Example:

Absolute Value Equation with Variables on Both Sides

$$|2x + 3| = -11x + 42$$

We begin by approaching this in analogy to the first problem above, $|x| = 10$. We saw that x could be either 10, or -10 . So we will assume in this case that $2x + 3$ can be either $-11x + 42$, or the negative of that, and solve both equations.

Problem One

- Equation:

$$2x + 3 = -11x + 42$$

- Equation:

$$13x + 3 = 42$$

- Equation:

$$13x = 39$$

- Equation:

$$x = 3$$

Problem Two

- Equation:

$$2x + 3 = -(-11x + 42)$$

• **Equation:**

$$2x + 3 = 11x - 42$$

• **Equation:**

$$-9x + 3 = -42$$

• **Equation:**

$$-9x = -45$$

• **Equation:**

$$x = 5$$

So we have two solutions: $x = 3$ and $x = 5$. Do they both work? Let's try them both.

Problem One

1. $|2(3) + 3| = -11(3) + 42.$

◦ $|9| = 9. \checkmark$

Problem Two

1. $|2(5) + 3| = -11(5) + 42.$

◦ $|13| = -13. \times$

We see in this case that the first solution, $x = 3$, worked; the second, $x = 5$, did not. So the only solution to this problem is $x = 3$.

However, there was no way of knowing that in advance. For such problems, the only approach is to solve them twice, and then **test both answers**. In some cases, both will work; in some cases, neither will work. In some cases, as in this one, one will work and the other will not.

Note: Whenever an absolute value equation has **variables on both sides**, you have to check your answer(s). Even if you do all the math perfectly, your answer(s) may not work.

OK, why is that? Why can you do all the math right and still get a wrong answer?

Remember that the problem $|x| = 10$ has two solutions, and $|x| = -10$ has none. We started with the problem $|2x + 3| = -11x + 42$. OK, which is that like? Is the right side of the equation like 10 or -10 ? If you think about it, you can convince yourself that **it depends on what x is**. After you solve, you may wind up with an x -value that makes the right side positive; that will work. Or, you may wind up with an x -value that makes the right side negative; that won't work. But you can't know until you get there.

Absolute Value Inequalities

This module introduces absolute value inequalities.

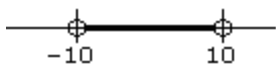
Here's one of my favorite problems:

$$|x| < 10$$

Having seen that the solution to $|x| = 10$ is $|x| = \pm 10$, many students answer this question $|x| < \pm 10$. However, this is not only wrong: it is, as discussed above, relatively meaningless. In order to approach this question you have to—you guessed it!—step back and think.

Here are two different, perfectly correct ways to look at this problem.

1. What numbers work? 4 works. -4 does too. 0 works. 13 doesn't work. How about -13 ? No: if $x = -13$ then $|x| = 13$, which is not less than 10. By playing with numbers in this way, you should be able to convince yourself that the numbers that work must be **somewhere between -10 and 10** . This is one way to approach the answer.
2. The other way is to think of absolute value as representing **distance from 0**. $|5|$ and $|-5|$ are both 5 because both numbers are 5 away from 0. In this case, $|x| < 10$ means “the distance between x and 0 is less than 10”—in other words, you are within 10 units of zero in either direction. Once again, we conclude that the answer must be **between -10 and 10** .



All numbers
whose
absolute value
is less than 10;
 $-10 < x < 10$

It is not necessary to use both of these methods; use whichever method is easier for you to understand.

More complicated absolute value problems should be approached in the same three steps as the equations discussed above: algebraically isolate the absolute value, then **think**, then algebraically solve for x . However, as illustrated above, the **think** step is a bit more complicated with inequalities than with equations.

Example:

Absolute Value Inequality

$$-3(|2x + 3| - 8) < -15$$

- **Step 1:** Algebraically isolate the absolute value
 - $|2x + 3| - 8 > 5$ (don't forget to switch the inequality when dividing by -3 !)
 - $|2x + 3| > 13$
- **Step 2:** Think!
 - As always, forget the $2x + 3$ in this step. The absolute value of **something** is greater than 13. What could the **something** be?
 - We can approach this in two ways, just as the previous absolute value inequality. The first method is trying numbers. We discover that all numbers greater than 13 work (such as 14, 15, 16)—their absolute values are greater than 13. Numbers less than -13 (such as $-14, -15, -16$) also have absolute values greater than 13. But in-between numbers, such as $-12, 0, 12$, do not work.
 - The other approach is to think of absolute value as representing distance to 0. The distance between **something** and 0 is greater than 13. So the **something** is more than 13 away from 0—in either direction.
 - Either way, we conclude that the **something** must be anything greater than 13, OR less than -13 !



The absolute value
of something is
greater than 13;
something < -13
OR
something > 13

- **Step 3:** Algebraically solve (both inequalities) for x
Equation:

$$\begin{aligned}2x + 3 < -13 & \text{ OR } 2x + 3 > 13 \\2x < -16 & \text{ OR } 2x > 10 \\x < -8 & \text{ OR } x > 5\end{aligned}$$



Any x -value
which is less
than -8 **or**
greater than
5 will make
the original
inequality
true;
 $x < -8$ OR
 $x > 5$

Many students will still resist the **think** step, attempting to figure out “the rules” that will always lead from the question to the answer. At first, it seems that memorizing a few rules won’t be too hard: “greater-than problems always lead to **OR** answers” and that kind of thing. But those rules will fail you when you hit a problem like the next one.

Example:

Absolute Value Inequality

$$|x - 3| + 10 > 7$$

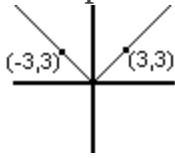
- **Step 1:** Algebraically isolate the absolute value
 - $|x - 3| > -3$
- **Step 2:** Think!
 - The absolute value of **something** is greater than -3 . What could the **something** be? 2 works. -2 also works. And 0. And 7. And -10 . And...hey! Absolute values are always positive, so the absolute value of anything is greater than -3 !

All numbers work

Graphing Absolute Values

This module describes how to graph absolute values.

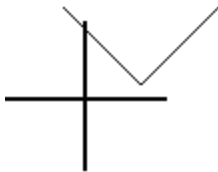
You can graph $y = |x|$ easily enough by plotting points. The characteristic V shape is illustrated below, with a couple of sample points highlighted.



$$y = |x|$$

Of course, this shape is subject to the same permutations as any other function! A few examples are given below.

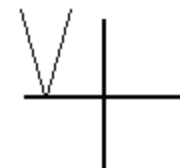
$|x - 5| + 1$
Moves right 5, up
1



$-|x| + 1$ Flips over x
-axis, moves up 1



$2|x + 5|$ 5 left,
vertically stretched

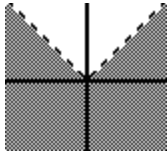


Inequalities and Absolute Value Concepts -- Graphing Inequalities

In general, the graph of an inequality is a **shaded area**.

Consider the graph $y = |x|$ shown above. Every point on that V-shape has the property that **its y -value is the absolute value of its x -value**. For instance, the point $(-3,3)$ is on the graph because 3 is the absolute value of -3 .

The inequality $y < |x|$ means **the y -value is less than the absolute value of the x -value**. This will occur anywhere **underneath** the above graph. For instance, the point $(-3,1)$ meets this criterion; the point $(-3,4)$ does not. If you think about it, you should be able to convince yourself that all points **below** the above graph fit this criterion.



$$y < |x|$$

The dotted line indicates that the graph $y = |x|$ is not actually a part of our set. If we were graphing $y \leq |x|$ the line would be complete, indicating that those points would be part of the set.

"Piecewise Functions" and Absolute Value

This module introduces piecewise functions for the purpose of understanding absolute value equations.

What do you get if you put a positive number into an absolute value?

Answer: you get that same number back. $|5| = 5$. $|\pi| = \pi$. And so on. We can say, as a generalization, that $|x| = x$; but **only if x is positive**.

OK, so, what happens if you put a **negative** number into an absolute value?

Answer: you get that same number back, but made positive. OK, how do you **make** a negative number positive? Mathematically, you **multiply it by -1** . $|-5| = -(-5) = 5$. $|-\pi| = -(-\pi) = \pi$. We can say, as a generalization, that $|x| = -x$; but **only if x is negative**.

So the absolute value function can be defined like this.

Equation:

The “Piecewise” Definition of Absolute Value

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

If you’ve never seen this before, it looks extremely odd. If you try to pin that feeling down, I think you’ll find this looks odd for some combination of these three reasons.

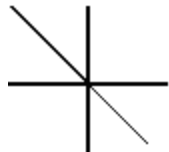
1. The whole idea of a “piecewise function”—that is, a function which is defined differently on different domains—may be unfamiliar. Think about it in terms of the function game. Imagine getting a card that says “If you are given a positive number or 0, respond with the same number you were given. If you are given a negative number, multiply it by -1 and give that back.” This is one of those “can a function **do that?**” moments. Yes, it can—and, in fact, functions defined in this “piecewise manner” are more common than you might think.
2. The $-x$ looks suspicious. “I thought an absolute value could never **be** negative!” Well, that’s right. But if x is negative, then $-x$ is positive. Instead of thinking of the $-x$ as “negative x ” it may help to think of it as “change the sign of x .”

3. Even if you get past those objections, you may feel that we have taken a perfectly ordinary, easy to understand function, and redefined it in a terribly complicated way. Why bother?

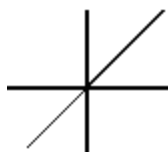
Surprisingly, the piecewise definition makes many problems **easier**. Let's consider a few graphing problems.

You already know how to graph $y = |x|$. But you can explain the V shape very easily with the piecewise definition. On the right side of the graph (where $x \geq 0$), it is the graph of $y = x$. On the **left** side of the graph (where $x < 0$), it is the graph of $y = -x$.

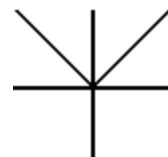
$y = -x$ The whole graph is shown, but the only part we care about is on the left, where $x < 0$



$y = x$ The whole graph is shown, but the only part we care about is on the right, where $x \geq 0$



$y = |x|$ Created by putting together the relevant parts of the other two graphs.

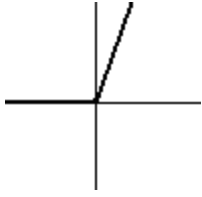


Still, that's just a new way of graphing something that we already knew how to graph, right? But now consider this problem: graph $y = x + |x|$. How do we approach that? With the piecewise definition, it becomes a snap.

Equation:

$$x + |x| = \begin{cases} x + x = 2x & x \geq 0 \\ x + (-x) = 0 & x < 0 \end{cases}$$

So we graph $y = 2x$ on the right, and $y = 0$ on the left. (You may want to try doing this in three separate drawings, as I did above.)



$$y = x + |x|$$

Our final example requires us to use the piecewise definition of the absolute value for both x and y .

Example:

Graph $|x| + |y| = 4$

We saw that in order to graph $|x|$ we had to view the left and right sides separately. Similarly, $|y|$ divides the graph **vertically**.

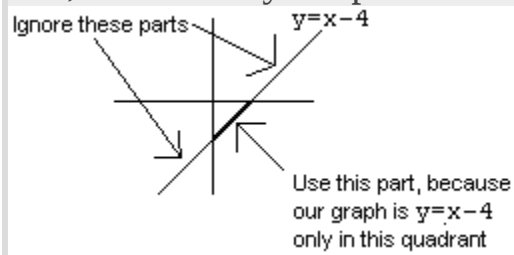
- On top, where $y \geq 0$, $|y| = y$.
- Where $y < 0$, on the bottom, $|y| = -y$.

Since this equation has **both** variables under absolute values, we have to divide the graph both horizontally and vertically, which means we look at **each quadrant separately**. $|x| + |y| = 4$

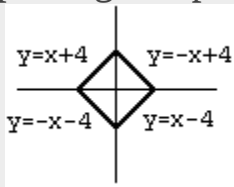
Second Quadrant	First Quadrant
$x \leq 0$, so $ x = -x$	$x \geq 0$, so $ x = x$
$y \geq 0$, so $ y = y$	$y \geq 0$, so $ y = y$

$(-x) + y = 4$	$x + y = 4$
$y = x + 4$	$y = -x + 4$
Third Quadrant	Fourth Quadrant
$x \leq 0$, so $ x = -x$	$x \geq 0$, so $ x = x$
$y \leq 0$, so $ y = -y$	$y \leq 0$, so $ y = -y$
$(-x) + (-y) = 4$	$x + (-y) = 4$
$y = -x - 4$	$y = x - 4$

Now we graph each line, but only in its respective quadrant. For instance, in the fourth quadrant, we are graphing the line $y = x - 4$. So we draw the line, but use only the part of it that is in the fourth quadrant.



Repeating this process in all four quadrants, we arrive at the proper graph.



$$|x| + |y| = 4$$

Simultaneous Equations Concepts -- Distance, Rate and Time

If you travel 30 miles per hour for 4 hours, how far do you go? A little common sense will tell you that the answer is 120 miles.

This relationship is captured in the following equation:

$d = rt$ where...

- d is distance traveled (sometimes the letter x is used instead, for position)
- r is the rate, or speed (sometimes the letter v is used, for velocity)
- t is the time

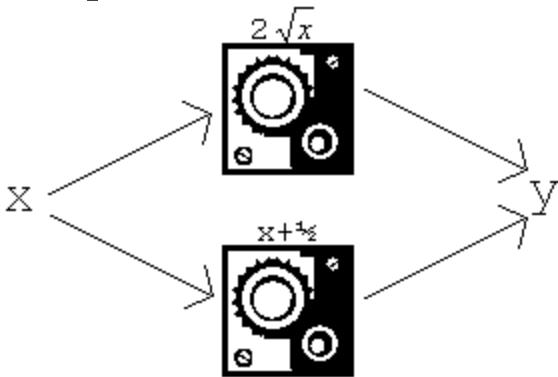
This is presented here because it forms the basis for many common simultaneous equations problems.

Simultaneous Equations Concepts -- Simultaneous Equations by Graphing

Consider the equation $y = 2\sqrt{x}$. How many (x,y) pairs are there that satisfy this equation? Answer: $(0,0)$, $(1,2)$, $(4,4)$, and $(9,6)$ are all solutions; and there is an **infinite number of other solutions**. (And don't forget non-integer solutions, such as $(,1)$!)

Now, consider the equation $y = x + \frac{1}{2}$. How many pairs satisfy **this** equation? Once again, an infinite number. Most equations that relate two variables have an infinite number of solutions.

To consider these two equations “simultaneously” is to ask the question: what (x,y) pairs make **both equations true**? To express the same question in terms of functions: what values can you hand the functions $2\sqrt{x}$ and $x + \frac{1}{2}$ that will make these two functions produce the **same answer**?

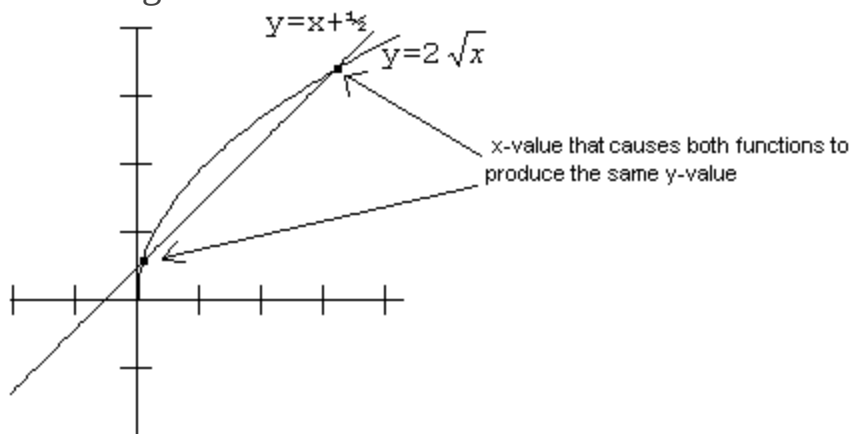


What number goes into **both functions** and makes them give the **same answer**? Is there even such a number? Is there more than one such number?

At first glance, it is not obvious how to approach such a question-- it is not even obvious how many answers there will be.

One way to answer such a question is by graphing. Remember, the graph of $y = 2\sqrt{x}$ is the set of all points that satisfy that relationship; and the graph of $y = x + \frac{1}{2}$ is the set of all points that satisfy that relationship. So **the intersection(s) of these two graphs is the set of all points that satisfy both relationships.**

How can we graph these two? The second one is easy: it is a line, already in $y = mx + b$ format. The y -intercept is $\frac{1}{2}$ and the slope is 1. We can graph the first equation by plotting points; or, if you happen to know what the graph of $y = \sqrt{x}$ looks like, you can stretch the graph vertically to get $y = 2\sqrt{x}$, since all the y -values will double. Either way, you wind up with something like this:



We can see that there are two points of intersection. One occurs when x is barely greater than 0 (say, $x = 0.1$), and the other occurs at approximately $x = 3$. There will be no more points of intersection after this, because the line will rise faster than the curve.

Exercise:

Problem: $y = 2\sqrt{x}$

$$y = x + \frac{1}{2}$$

Solution:

From graphing...

$$x = 0.1, x = 3$$

Graphing has three distinct advantages as a method for solving simultaneous equations.

1. It works on any type of equations.
2. It tells you **how many** solutions there are, as well as what the solutions are.
3. It can help give you an **intuitive feel** for why the solutions came out the way they did.

However, graphing also has two **disadvantages**.

1. It is time-consuming.
2. It often yields solutions that are **approximate**, not exact—because you find the solutions by simply “eyeballing” the graph to see where the two curves meet.

For instance, if you plug the number 3 into both of these functions, will you get the same answer?

$$3 \rightarrow 2\sqrt{x} \rightarrow 2\sqrt{3} \approx 3.46$$

$$3 \rightarrow x + \frac{1}{2} \rightarrow 3.5$$

Pretty close! Similarly, $2\sqrt{.1} \approx 0.632$, which is quite close to 0.6. But if we want more exact answers, we will need to draw a much more exact graph, which becomes **very** time-consuming. (Rounded to three decimal places, the actual answers are 0.086 and 2.914.)

For more exact answers, we use analytic methods. Two such methods will be discussed in this chapter: **substitution** and **elimination**. A third method will be discussed in the section on Matrices.

Simultaneous Equations Concepts -- Substitution

Here is the algorithm for substitution.

1. Solve one of the equations for one variable.
2. Plug this variable into the other equation.
3. Solve the second equation, which now has only one variable.
4. Finally, use the equation you found in step (1) to find the other variable.

Example:

Solving Simultaneous Equations by Substitution

$$3x + 4y = 1$$

$$2x - y = 8$$

1. The easiest variable to solve for here is the y in the second equation.

- $-y = -2x + 8$
- $y = 2x - 8$

2. Now, we plug that into the **other** equation:

- $3x + 4(2x - 8) = 1$

3. We now have an equation with only x in it, so we can solve for x .

- $3x + 8x - 32 = 1$
- $11x = 33$
- $x = 3$

4. Finally, we take the equation from step (1), $y = 2x - 8$, and use it to find y .

- $y = 2(3) - 8 = -2$

So $(3, -2)$ is the solution. You can confirm this by plugging this pair into both of the original equations.

Why does substitution work?

We found in the first step that $y = 2x - 8$. This means that y and $2x - 8$ are **equal** in the sense that we discussed in the first chapter on functions—they will always be the same number, in these equations—they are the **same**. This gives us permission to simply replace one with the other, which is what we do in the second (“substitution”) step.

Simultaneous Equations Concepts -- Elimination

Here is the algorithm for elimination.

1. Multiply one equation (or in some cases both) by some number, so that the two equations have the **same coefficient** for one of the variables.
2. Add or subtract the two equations to make that variable go away.
3. Solve the resulting equation, which now has only one variable.
4. Finally, plug back in to find the other variable.

Example:

Solving Simultaneous Equations by Elimination

$$3x + 4y = 1$$

$$2x - y = 8$$

- **1**The first question is: how do we get one of these variables to have the **same coefficient in both equations**? To get the x coefficients to be the same, we would have to multiply the top equation by 2 and the bottom by 3. It is much easier with y ; if we simply multiply the bottom equation by 4, then the two y values will both be multiplied by 4.
 - $3x + 4y = 1$
 - $8x - 4y = 32$
- **2**Now we either **add** or **subtract** the two equations. In this case, we have $4y$ on top, and $-4y$ on the bottom; so if we add them, they will cancel out. (If the bottom had $+4y$ we would have to subtract the two equations to get the " y "s to cancel.)
 - $11x + 0y = 33$
- **3-4** Once again, we are left with only one variable. We can solve this equation to find that $x = 3$ and then plug back in to either of the original equations to find $y = -2$ as before.

Why does elimination work?

As you know, you are always allowed to do the **same thing** to both sides of an equation. If an equation is true, it will still be true if you add 4 to both sides, multiply both sides by 6, or take the square root of both sides.

Now—consider, in the second step above, what we did to the equation $3x + 4y = 1$. We **added** something to both sides of this equation. What did we add? On the left, we added $8x - 4y$; on the right, we added 32. It seems that we have done something **different** to the two sides.

However, the second equation gives us a guarantee that these two quantities, $8x - 4y$ and 32, are in fact **the same as each other**. So by adding $8x - 4y$ to the left, and 32 to the right, we really have done exactly the same thing to both sides of the equation $3x + 4y = 1$.

Special Cases

This module discusses concepts related to simultaneous equations.

Consider the two equations:

Equation:

$$2x + 3y = 8$$

Equation:

$$4x + 6y = 3$$

Suppose we attempt to solve these two equations by elimination. So, we double the first equation and subtract, and the result is:

Equation:

$$4x + 6y = 16$$

$$4x + 6y = 3$$

$$0 = 13$$

Hey, what happened? 0 does not equal 13, no matter what x is.

Mathematically, we see that these two equations have **no** simultaneous solution. You asked the question “When will both of these equations be true?” And the math answered, “Hey, buddy, not until 0 equals 13.”

No solution.

Now, consider these equations:

Equation:

$$2x + 3y = 8$$

$$4x + 6y = 16$$

Once again, we attempt elimination, but the result is different:

Equation:

$$2x + 3y = 8$$

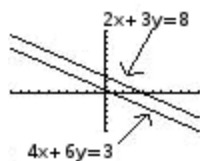
$$4x + 6y = 16$$

$$0 = 0$$

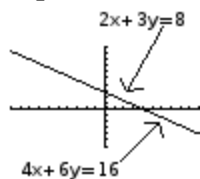
What happened **that** time? $0 = 0$ no matter what x is. Instead of an equation that is always false, we have an equation that is always true. Does that mean these equations work for **any** x and y ? Clearly not: for instance, $(1,1)$ does not make either equation true. What this means is that the two equations are the same: any pair that solves one will also solve the other. There is an infinite number of solutions.

Infinite number of solutions.

All of this is much easier to understand graphically! Remember that one way to solve simultaneous equations is by graphing them and looking for the intersection. In the first case, we see that original equations represented two **parallel lines**. There is no point of intersection, so there is no simultaneous equation.



In the second case, we see that the original equations represented the **same line, in two different forms**. Any point on the line is a solution to both equations.



Note: If you solve an equation and get a mathematical impossibility such as $0 = 13$, there is no solution. If you get a mathematical tautology such as $0 = 0$, there is an infinite number of solutions.

Simultaneous Equations Concepts -- Word Problems

Many students approach math with the attitude that “I can do the equations, but I’m just not a ‘word problems’ person.” No offense, but that’s like saying “I’m pretty good at handling a tennis racket, as long as there’s no ball involved.” The only point of handling the tennis racket is to hit the ball. The only point of math equations is to solve problems. So if you find yourself in that category, try this sentence instead: “I’ve never been good at word problems. There must be something about them I don’t understand, so I’ll try to learn it.”

Actually, many of the key problems with word problems were discussed in the very beginning of the “Functions” unit, in the discussion of variable descriptions. So this might be a good time to quickly re-read that section. If you can correctly identify the variables, you’re half-way through the hard part of a word problem. The other half is **translating the sentences of the problem into equations** that use those variables.

Let’s work through an example, very carefully.

Example:

Simultaneous Equation Word Problem

A roll of dimes and a roll of quarters lie on the table in front of you. There are three more quarters than dimes. But the quarters are worth three times the amount that the dimes are worth. How many of each do you have?

1. Identify and label the variables.

- There are actually two different, valid ways to approach this problem. You could make a variable that represents the number of dimes; or you could have a variable that represents the value of the dimes. Either way will lead you to the right answer. However, it is vital to know which one you’re doing! If you get confused half-way through the problem, you will end up with the wrong answer.

Let's try it this way:

d is the number of dimes

q is the number of quarters

2. Translate the sentences in the problem into equations.

- "There are three more quarters than dimes" $\rightarrow q = d + 3$
- "The quarters are worth three times the amount that the dimes are worth" $\rightarrow 25q = 3(10d)$
- This second equation relies on the fact that if you have q quarters, they are worth a total of $25q$ cents.

3. Solve.

- We can do this by elimination or substitution. Since the first equation is already solved for q , I will substitute that into the second equation and then solve.

$$25(d + 3) = 3(10d)$$

$$25d + 75 = 30d$$

$$75 = 5d$$

$$d = 15$$

$$q = 18$$

So, did it work? The surest check is to go all the way back to the original problem—not the equations, but the words. We have concluded that there are 15 **dimes** and **18** quarters.

“There are three more quarters than dimes.” ✓

“The quarters are worth three times the amount that the dimes are worth.”

→ Well, the quarters are worth $18 \cdot 25 = \$4.50$. The dimes are worth $15 \cdot 10 = \$1.50$. ✓

Using Letters as Numbers

This module shows how it can be helpful, on occasion, to use letters as numbers in order to quickly find solutions to variations on a simultaneous equation problem.

Toward the end of this chapter, there are some problems in substitution and elimination where **letters are used in place of numbers**. For instance, consider the following problem:

Equation:

$$2y - ax = 7$$

Equation:

$$4y + 3ax = 9$$

What do we do with those "a"s? Like any other variable, they simply represent an unknown number. As we solve for x , we will simply leave a as a variable.

This problem lends itself more naturally to elimination than to substitution, so I will double the top equation and then subtract the two equations and solve.

Equation:

$$\begin{array}{r} 4y - 2ax = 14 \\ -(4y + 3ax = 9) \\ \hline 0y - 5ax = 5 \end{array}$$

Equation:

$$x = \frac{5}{-5a} = \frac{-1}{a}$$

As always, we can solve for the second variable by plugging into either of our original equations.

Equation:

$$2y - a\left(\frac{-1}{a}\right) = 7$$

Equation:

$$2y + 1 = 7$$

Equation:

$$y = 3$$

There is no new math here, just elimination. The real trick is not to be spooked by the a , and do the math just like you did before.

And what does that **mean**? It means we have found a solution that works for those two equations, regardless of a . We can now solve the following three problems (and an infinite number of others) without going through the hard work.

If $a = 5$,	If $a = 10$,	If $a = -3$,
The original equations become:	The original equations become:	The original equations become:
$2y - 5x = 7$ $4y + 15x = 9$	$2y - 10x = 7$ $4y + 30x = 9$	$2y + 3x = 7$ $4y - 9x = 9$
And the solution is:	And the solution is:	And the solution is:

If $a = 5$,	If $a = 10$,	If $a = -3$,
$x = \frac{-1}{5}, y = 3$	$x = \frac{-1}{10}, y = 3$	$x = \frac{1}{3}, y = 3$

The whole point is that I did not have to **solve** those three problems—by elimination, substitution, or anything else. All I had to do was plug a into the general answer I had already found previously. If I had to solve a hundred such problems, I would have saved myself a great deal of time by going through the hard work once to find a general solution!

Mathematicians use this trick all the time. If they are faced with many similar problems, they will attempt to find a **general** problem that encompasses all the **specific** problems, by using variables to replace the numbers that change. You will do this in an even more general way in the text, when you solve the “general” simultaneous equations where **all** the numbers are variables. Then you will have a formula that you can plug any pair of simultaneous equations into to find the answer at once. This formula would also make it very easy, for instance, to program a computer to solve simultaneous equations (computers are terrible at figuring things out, but they’re great at formulas).

Multiplying Binomials

This module teaches about multiplying binomials. Specifically about common patterns that can be memorized and using the "FOIL" method.

The following three formulae should be memorized.

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

Equation:

$$(x + a)(x - a) = x^2 - a^2$$

It is important to have these three formulae on the top of your head. It is also nice to be able to show **why** these formulae work, for instance by using FOIL. But the most important thing of all is knowing **what these three formulae mean, and how to use them.**

These three are all "algebraic generalizations," as discussed in the first unit on functions. That is, they are equations that hold true for **any** values of x and a . It may help if you think of the second equation above as standing for:

$(\text{Anything} - \text{Anything Else})^2 = \text{Anything}^2 - 2(\text{Anything Else})^2$ For instance, suppose the **Anything** (or x) is 5, and the **Anything Else** (or a) is 3.

Example:

$$(x - a)^2 = x^2 - 2ax + a^2, \text{ when } x = 5, a = 3.$$

- $5 - 3^2 \stackrel{?}{=} 5^2 - 2(3)(5) + 3^2$
- $2^2 \stackrel{?}{=} 25 - 30 + 9$
- $4 = 4$

It worked! Now, let's leave the **Anything** as x , but play with different values of a .

Example:

More examples of $(x - a)^2 = x^2 - 2ax + a^2$

Equation:

$$\begin{aligned}
 a = 1 : (x - 1)^2 &= x^2 - 2x + 1 \\
 a = 2 : (x - 2)^2 &= x^2 - 4x + 4 \\
 a = 3 : (x - 3)^2 &= x^2 - 6x + 9 \\
 a = 5 : (x - 5)^2 &= x^2 - 10x + 25 \\
 a = 10 : (x - 10)^2 &= x^2 - 20x + 100
 \end{aligned}$$

Once you've seen a few of these, the pattern becomes evident: the number **doubles** to create the middle term (the coefficient of x), and **squares** to create the final term (the number).

The hardest thing about this formula is **remembering to use it**. For instance, suppose you are asked to expand:

Equation:

$$(2y - 6)^2$$

There are three ways you can approach this.

$(2y - 6)^2$, computed three different ways		
Square each term	FOIL	Using the formula above
$ \begin{aligned} &(2y - 6)^2 \\ &(2y)^2 - 2(6)(2y) + 6^2 \\ &4y^2 - 24y + 36 \end{aligned} $	$ \begin{aligned} &(2y - 6)(2y - 6) \\ &(2y)(2y) - (2y)6 - (2y)6 + 36 \\ &4y^2 - 12y - 12y + 36 \\ &4y^2 - 24y + 36 \end{aligned} $	$ \begin{aligned} &(2y - 6)^2 \\ &(2y)^2 - 2(6)(2y) + 6^2 \\ &4y^2 - 24y + 36 \end{aligned} $

Did it work? If a formula is true, it should work for any y -value; let's test each one with $y = 5$. (Note that the second two methods got the same answer, so we only need to test that once.)

--	--

$$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 36$$

$$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4y^2 - 36$$

$$(10 - 6)^2 \stackrel{?}{=} 100 - 36$$

$$4^2 \stackrel{?}{=} 64 \times$$

$$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 24y + 36$$

$$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4(5)^2 - 24 \cdot 5 + 36$$

$$(10 - 6)^2 \stackrel{?}{=} 100 - 120 + 36$$

$$4^2 \stackrel{?}{=} 16 \checkmark$$

We conclude that squaring each term individually does not work. The other two methods both give the same answer, which works.

The first method is the easiest, of course. And it **looks** good. $(2y)^2$ is indeed $4y^2$. And 6^2 is indeed 36. But as you can see, it led us to a **false answer**—an algebraic generalization that did not hold up.

I just can't stress this point enough. It sounds like a detail, but it causes errors all through Algebra II and beyond. **When you're adding or subtracting things, and then squaring them, you can't just square them one at a time.** Mathematically, $(x + a)^2 \neq x^2 + a^2$. You can confirm this with numbers all day. $(7 + 3)^2 = 100$, but $7^2 + 3^2 = 58$. They're not the same.

So that leaves the other two methods. FOIL will never lead you astray. But the third approach, the formula, has three distinct advantages.

1. The formula is faster than FOIL.
2. Using these formulae is a specific case of the vital mathematical skill of using **any** formula—learning how to plug numbers and variables into some equation that you've been given, and therefore understanding the abstraction that formulae represent.
3. Before this unit is done, we will be completing the square, which requires running that particular formula **backward**—which you cannot do with FOIL.

Factoring

This module discusses how to solve quadratic equations by factoring.

When we multiply, we put things together: when we factor, we pull things apart. Factoring is a critical skill in simplifying functions and solving equations.

There are four basic types of factoring. In each case, I will start by showing a multiplication problem—then I will show how to use factoring to **reverse** the results of that multiplication.

“Pulling Out” Common Factors

This type of factoring is based on the **distributive property**, which (as you know) tells us that:

Equation:

$$2x(4x^2 - 7x + 3) = 8x^3 - 14x^2 + 6x$$

When we factor, we do that in reverse. So we would start with an expression such as $8x^3 - 14x^2 + 6x$ and say “Hey, **every one** of those terms is divisible by 2. Also, **every one** of those terms is divisible by x . So we “factor out,” or “pull out,” a $2x$.”

Equation:

$$8x^3 - 14x^2 + 6x = 2x(__ - __ + __)$$

For each term, we see what happens when we **divide** that term by $2x$. For instance, if we divide $8x^3$ by $2x$ the answer is $4x^2$. Doing this process for each term, we end up with:

Equation:

$$8x^3 - 14x^2 + 6x = 2x(4x^2 - 7x + 3)$$

As you can see, this is just what we started with, but in reverse. However, for many types of problems, this factored form is easier to work with.

As another example, consider $6x + 3$. The common factor in this case is 3. When we factor a 3 out of the $6x$, we are left with $2x$. When we factor a 3 out of the 3, we are left with...what? Nothing? No, we are left with 1, since we are **dividing** by 3.

Equation:

$$6x + 3 = 3(2x + 1)$$

There are two key points to take away about this kind of factoring.

1. This is the simplest kind of factoring. Whenever you are trying to factor a complicated expression, **always begin** by looking for **common factors** that you can pull out.
2. A common factor must be common to **all** the terms. For instance, $8x^3 - 14x^2 + 6x + 7$ has no common factor, since the last term is not divisible by either 2 or x .

Factoring Perfect Squares

The second type of factoring is based on the “squaring” formulae that we started with:

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

For instance, if we see $x^2 + 6x + 9$, we may recognize the signature of the first formula: the middle term is three **doubled**, and the last term is three

squared. So this is $(x + 3)^2$. Once you get used to looking for this pattern, it is easy to spot.

Equation:

$$x^2 + 10x + 25 = (x + 5)^2$$

Equation:

$$x^2 + 2x + 1 = (x + 1)^2$$

And so on. If the middle term is **negative**, then we have the second formula:

Equation:

$$x^2 - 8x + 16 = (x - 4)^2$$

Equation:

$$x^2 - 14x + 49 = (x - 7)^2$$

This type of factoring only works if you have **exactly this case**: the middle number is something **doubled**, and the last number is that same something **squared**. Furthermore, although the **middle** term can be either positive or negative (as we have seen), the **last** term cannot be negative.

All this may make it seem like such a special case that it is not even worth bothering about. But as you will see with “completing the square” later in this unit, this method is very general, because even if an expression does not look like a perfect square, you can usually **make** it look like one if you want to—and if you know how to spot the pattern.

The Difference Between Two Squares

The third type of factoring is based on the third of our basic formulae:

Equation:

$$(x + a)(x - a) = x^2 - a^2$$

You can run this formula in reverse whenever you are **subtracting two perfect squares**. For instance, if we see $x^2 - 25$, we recognize that both x^2 and 25 are perfect squares. We can therefore factor it as $(x + 5)(x - 5)$. Other examples include:

Example:

- $x^2 - 64 = (x + 8)(x - 8)$
- $16y^2 - 49 = (4y + 7)(4y - 7)$
- $2x^2 - 18 = 2(x^2 - 9) = 2(x + 3)(x - 3)$

And so on. Note that, in the last example, we begin by **pulling out** a 2, and we are then left with two perfect squares. This is an example of the rule that you should always **begin by pulling out common factors** before you try anything else!

It is also important to note that you cannot factor the sum of two squares. $x^2 + 4$ is a perfectly good function, but it cannot be factored.

Brute Force, Old-Fashioned, Bare-Knuckle, No-Holds-Barred Factoring

In this case, the multiplication that we are reversing is just FOIL. For instance, consider:

Equation:

$$(x + 3)(x + 7) = x^2 + 3x + 7x + 21 = x^2 + 10x + 21$$

What happened? The 3 and 7 **added** to yield the middle term (10), and **multiplied** to yield the final term (21). We can generalize this as:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

The point is, if you are given a problem such as $x^2 + 10x + 21$ to factor, you look for two numbers that **add** up to 10, and **multiply** to 21. And how do you find them? There are a lot of pairs of numbers that add up to 10, but relatively few that multiply to 21. So you start by looking for factors of 21.

Below is a series of examples. Each example showcases a different aspect of the factoring process, so I would encourage you not to skip over any of them: try each problem yourself, then take a look at what I did.

If you are uncomfortable with factoring, the best practice you can get is to multiply things out. In each case, look at the final answer I arrive at, and multiply it with FOIL. See that you get the problem I started with. Then look back at the steps I took and see how they led me to that answer. The steps will make a lot more sense if you have done the multiplication already.

Exercise:

Problem: Factor $x^2 + 11x + 18$

$$(x + \underline{\quad})(x + \underline{\quad})$$

Solution:

What **multiplies** to 18? $1 \cdot 18$, or $2 \cdot 9$, or $3 \cdot 6$.

Which of those **adds** to 11? $2 + 9$.

$$(x + 2)(x + 9)$$

Note: Start by listing all factors of the third term. Then see which ones add to give you the middle term you want.

Exercise:

Problem: Factor $x^2 - 13x + 12$

$$(x + \underline{\quad})(x + \underline{\quad})$$

Solution:

What **multiplies** to 12? $1 \cdot 12$, or $2 \cdot 6$, or $3 \cdot 4$

Which of those **adds** to 13? $1 + 12$

$$(x - 1)(x - 12)$$

Note: If the middle term is negative, it doesn't change much: it just makes both numbers negative. If this had been $x^2 + 13x + 12$, the process would have been the same, and the answer would have been $(x + 1)(x + 12)$.

Exercise:

Problem: Factor $x^2 + 12x + 24$

$$(x + \underline{\quad})(x + \underline{\quad})$$

Solution:

What **multiplies** to 24? $1 \cdot 24$, or $2 \cdot 12$, or $3 \cdot 8$, or $4 \cdot 6$

Which of those **adds** to 12? None of them.

It can't be factored. It is "prime."

Note: Some things can't be factored. Many students spend a long time fighting with such problems, but it really doesn't have to take long. Try all the possibilities, and if none of them works, it can't be factored.

Exercise:

Problem: Factor $x^2 + 2x - 15$

$$(x + \underline{\quad})(x + \underline{\quad})$$

Solution:

What **multiplies** to 15? $1 \cdot 15$, or $3 \cdot 5$

Which of those **subtracts** to 2? $5-3$

$$(x + 5)(x - 3)$$

Note: If the **last** term is negative, that changes things! In order to **multiply** to -15 , the two numbers will have to have different signs—one negative, one positive—which means they will **subtract** to give the middle term. Note that if the **middle** term were negative, that wouldn't change the process: the final answer would be reversed, $(x + 5)(x - 3)$. This fits the rule that we saw earlier—changing the sign of the middle term changes the answer a bit, but not the process.

Exercise:

Problem: Factor $2x^2 + 24x + 72$

Solution:

$$2(x^2 + 12x + 36)$$

$$2(x + 6)^2$$

Note: Never forget, **always** start by looking for common factors to pull out. Then look to see if it fits one of our formulae. Only after trying all that do you begin the FOIL approach.

Exercise:

Problem: Factor $3x^2 + 14x + 16$

$$(3x + \underline{\quad})(x + \underline{\quad})$$

Solution:

What **multiplies** to 16? $1 \cdot 16$, or $2 \cdot 8$, or $4 \cdot 4$

Which of those adds to 14 **after tripling one number**? $8 + 3 \cdot 2$

$$(3x + 8)(x + 2)$$

Note: If the x^2 has a coefficient, and if you can't pull it out, the problem is trickier. In this case, we know that the factored form will look like $(3x + \underline{\quad})(x + \underline{\quad})$ so we can see that, when we multiply it back, one of those numbers—the one on the right—will be tripled, before they add up to the middle term! So you have to check the number pairs to see if any work that way.

Checking Your Answers

There are two different ways to check your answer after factoring: multiplying back, and trying numbers.

Example:

1. **Problem:** Factor $40x^3 - 250x$

- $10x(4x - 25)$ **First, pull out the common factor**
- $10x(2x + 5)(2x - 5)$ **Difference between two squares**

2. **So, does $40x^3 - 250x = 10x(2x + 5)(2x - 5)$? First let's check by multiplying back.**

- $10x(2x + 5)(2x - 5)$
- $= (20x^2 + 50x)(2x - 5)$ **Distributive property**
- $= 40x^3 - 100x^2 + 100x^2 - 250x$ **FOIL**
- $= 40x^3 - 250x$

3. **Check by trying a number. This should work for any number. I'll use $x = 7$ and a calculator.**

- $40x^3 - 250x = 10x(2x + 5)(2x - 5)$
- $40(7)^3 - 250(7) = 10(7)(2 \cdot 7 + 5)(2 \cdot 7 - 5)$
- $11970 = 11970$

I stress these methods of checking answers, not just because checking answers is a generally good idea, but because they reinforce key concepts. The first method reinforces the idea that **factoring is multiplication done backward**. The second method reinforces the idea of algebraic generalizations.

Quadratic Concepts -- Solving Quadratic Equations by Factoring

Consider the equation $4x^2 + 14x - 60 = 0$. This is **not** an algebraic generalization, but an equation to be solved for x : that is, it asks the question “What x value, or values, will make this equation true?” We will be solving such equations in three different ways. The fastest and easiest is by factoring.

Using the techniques discussed above, we can rewrite this problem as follows. (Try it for yourself!)

$$4x^2 + 14x - 60 = 0 \text{ **Original form**}$$

$$2(2x - 5)(x + 6 = 0) \text{ **Factored form**}$$

The second form may look more complicated than what we started with. But consider what this equation says. There are three numbers: 2, $2x - 5$, and $x + 6$. The equation says that when you multiply these three numbers, you get 0. Ask yourself this crucial question: **How can you multiply numbers and get the answer 0?**

The only way it can happen is if one of the numbers is 0. Take a moment to convince yourself of this: if several numbers **multiply** to give 0, one of those numbers must be 0.

So we have three possibilities.

$2 = 0$	$2x - 5 = 0$	$x + 6 = 0$
(it just isn't)	$x = 2\frac{1}{2}$	$x = -6$

The moral of the story is: when a quadratic equation is factored, it can be solved easily. In this case, the equation $4x^2 + 14x - 60 = 0$ has two valid solutions, $x = 2\frac{1}{2}$ and $x = -6$.

Consider this example:

Equation:

$$x^2 - 9x + 20 = 6$$

A common mistake is to solve it like this.

Example:

$x^2 - 9x + 20 = 6$, solved **incorrectly**

- $(x - 4)(x - 5) = 6$
- $(x - 4) = 6$
 - $x = 10$ ✗
- $(x - 5) = 6$
 - $x = 11$ ✗

All looks good, doesn't it? The factoring was correct. But if you try $x = 10$ or $x = 11$ in the original equation, you will find that neither one works. What went wrong?

The factoring was correct, but the next step was wrong. Just because $(x - 4)(x - 5) = 6$ does not mean that either $(x - 4)$ or $(x - 5)$ has to be 6. There are lots of ways for two numbers to multiply to give 6. This trick only works for 0!

Example: $x^2 - 9x + 20 = 6$, solved **correctly**

- $x^2 - 9x + 14 = 0$
- $(x - 7)(x - 2) = 0$
- $(x - 7) = 0$
 - $x = 7$ ✓
- $(x - 2) = 0$
 - $x = 2$ ✓

You may want to confirm for yourself that these are the correct solutions.

Moral: When solving quadratic equations, always begin by moving **everything to one side of the equation**, leaving only a 0 on the other side. This is true regardless of which of the three methods you use.

Example: $x^2 + 14x + 49 = 0$

- $(x + 7)^2 = 0$
- $x = -7$

Moral: If the left side factors as a perfect square, the quadratic equation has only one solution.

Not all quadratic functions can be factored. This does not mean they have no solutions! If the function cannot be factored, we must use other means to

find the solutions.

Solving Quadratic Equations by Completing the Square

This module teaches the method of solving quadratic equations by completing the square.

Consider the equation:

Equation:

$$(x + 3)^2 = 16$$

We can solve this by analogy to the way that we approached absolute value problems. **something** squared is 16. So what could the **something** be? It could be 4. It could also be -4 . So the solution is:

- $x + 3 = 4$
 - $x = 1$
- $x + 3 = -4$
 - $x = -7$

These are the two solutions.

This simple problem leads to a completely general way of solving quadratic equations—because any quadratic equation can be put in a form like the above equation. The key is **completing the square** which, in turn, is based on our original two formulae:

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

As an example, consider the equation $x^2 + 10x + 21 = 0$. In order to make it fit one of the patterns above, we must replace the 21 with the correct number: a number such that $x^2 + 10x + \underline{\quad}$ is a perfect square. What number goes there? If you are familiar with the pattern, you know the answer right away. 10 is 5 **doubled**, so the number there must be 5 **squared**, or 25.

But how do we turn a 21 into a 25? We add 4, of course. And if we add 4 to **one** side of the equation, we have to add 4 to the **other** side of the equation. So the entire problem is worked out as follows:

Example:

Solving by Completing the Square (quick-and-dirty version)

Solve $x^2 + 10x + 21 = 0$

The problem.

$x^2 + 10x + 25 = 4$

Add 4 to both sides, so that the left side becomes a perfect square.

$(x + 5)^2 = 4$

Rewrite the perfect square.

$x + 5 = 2$
 $x = -3$

$x + 5 = -2$
 $x = -7$

If something-squared is 4, the something can be 2, or -2 . Solve both possibilities to find the two answers.

Thus, we have our two solutions.

Completing the square is more time-consuming than factoring: so whenever a quadratic equation can be factored, factoring is the preferred method. (In this case, we would have factored the original equation as $(x + 3)(x + 7)$ and gotten straight to the answer.) However, completing the square can be used on **any** quadratic equation. In the example below, completing the square is used to find two answers that would not have been found by factoring.

Example:

Solving by Completing the Square (showing all the steps more carefully)

$$\text{Solve } 9x^2 - 54x + 80 = 0$$

The problem.

$$9x^2 - 54x = -80$$

Put all the x terms on one side, and the number on the other

$$x^2 - 6x = -\frac{80}{9}$$

Divide both sides by the coefficient of x^2 (*see below)

$$x^2 - 6x + 9 = -\frac{80}{9} + \underline{9}$$

Add the same number (*see below) to both sides, so that the left side becomes a perfect square.

$$(x - 3)^2 = -\frac{80}{9} + \frac{81}{9} = \frac{1}{9}$$

Rewrite the perfect square.

Solving by Completing the Square (showing all the steps more carefully)

$$x - 3 = \frac{1}{3}$$

$$x = 3\frac{1}{3}$$

$$x - 3 = -\frac{1}{3}$$

$$x = 2\frac{2}{3}$$

If something-squared is $\frac{1}{9}$, the something can be $\frac{1}{3}$, or $-\frac{1}{3}$. Solve both possibilities to find the two answers.

Two steps in particular should be pointed out here.

In the third step, we divide both sides by 9. When completing the square, you do not want to have any coefficient in front of the term; if there is a number there, you divide it out. Fractions, on the other hand (such as the $-\frac{80}{9}$ in this case) do not present a problem. This is in marked contrast to factoring, where a coefficient in front of the x^2 can be left alone, but fractions make things nearly impossible.

The step after that is where we actually complete the square. $x^2 + 6x + \underline{\quad}$ will be our perfect square. How do we find what number we want? Start with the coefficient of x^2 (in this case, 6). **Take half of it, and square the result.** Half of 6 is 3, squared is 9. So we want a 9 there to create $x^2 + 6x + 9$ which can be simplified to $(x + 3)^2$.

$$x^2 - 6x + 9$$

Take half of this and put it here \rightarrow $(x - 3)^2$ \leftarrow Then square that and put it here

If the coefficient of x is an **odd number**, the problem becomes a little uglier, but the principle is the same. For instance, faced with:

Equation:

$$x^2 + 5x + \underline{\quad}$$

You would begin by taking half of 5 (which is $\frac{5}{2}$) and then squaring it:

$$x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$$

Another “completing the square” example, in which you cannot get rid of the square root at all, is presented in the worksheet “The Generic Quadratic Equation.”

One final note on completing the square: there are three different possible outcomes.

- If you end up with something like $(x - 3)^2 = 16$ you will find two solutions, since $x - 3$ can be either 4, or -4 . You will always have two solutions if the right side of the equation, after completing the square, is positive.
- If you end up with $(x - 3)^2 = 0$ then there is only one solution: x must be 3 in this example. If the right side of the equation is 0 after completing the square, there is only one solution.
- Finally, what about a negative number on the right, such as $(x - 3)^2 = -16$? Nothing squared can give a negative answer, so there is no solution.

The Quadratic Formula

This module discusses the quadratic formula.

In "Solving Quadratic Equations by Completing the Square" I talked about the common mathematical trick of solving a problem once, **using letters instead of numbers**, and then solving specific problems by plugging numbers into a general solution.

In the text, you go through this process for quadratic equations in general. The definition of a quadratic equation is any equation that can be written in the form:

Equation:

$$ax^2 + bx + c = 0$$

where $a \neq 0$. By completing the square on this generic equation, you arrive at the quadratic formula:

Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula can then be used to solve any quadratic equation, without having to complete the square each time. To see how this formula works, let us return to the previous problem:

Equation:

$$9x^2 - 54x + 80 = 0$$

In this case, $a = 9$, $b = -54$, and $c = 80$. So the quadratic formula tells us that the answers are:

Equation:

$$x = \frac{-(-54) \pm \sqrt{(-54)^2 - 4(9)(80)}}{2(9)}$$

We'll use a calculator here rather than squaring 54 by hand...

Equation:

$$x = \frac{54 \pm \sqrt{2916 - 2880}}{18} = \frac{54 \pm \sqrt{36}}{18} = \frac{54 \pm 6}{18} = \frac{9 \pm 1}{3}$$

So we find that the two answers are $\frac{10}{3}$ and $\frac{8}{3}$, which are the same answers we got by completing the square.

Using the quadratic formula is usually faster than completing the square, though still slower than factoring. So, in general, try to factor first: if you cannot factor, use the quadratic formula.

So why do we learn completing the square? Two reasons. First, completing the square is how you **derive** the quadratic formula. Second, completing the square is vital to **graphing** quadratic functions, as you will see a little further on in the chapter.

Quadratic Concepts -- Different Types of Solutions to Quadratic Equations

The heart of the quadratic formula is the part under the square root: $b^2 - 4ac$. This part is so important that it is given its own name: the **discriminant**. It is called this because it **discriminates** the different types of solutions that a quadratic equation can have.

Common Error

Students often think that the discriminant is $\sqrt{b^2 - 4ac}$. But the discriminant is **not the square root**, it is the part that is **under** the square root:

Equation:

$$\text{Discriminant} = b^2 - 4ac$$

It can often be computed quickly and easily without a calculator.

Why is this quantity so important? Consider the above example, where the discriminant was 36. This means that we wound up with ± 6 in the numerator. So the problem had two different, rational answers: $2\frac{2}{3}$ and $3\frac{1}{3}$.

Now, consider $x^2 + 3x + 1 = 0$. In this case, the discriminant is $3^2 - 4(1)(1) = 5$. We will end up with $\pm\sqrt{5}$ in the numerator. There will still be two answers, but they will be **irrational**—they will be impossible to express as a fraction without a square root.

$4x^2 - 20x + 25 = 0$. Now the discriminant is $20^2 - 4(4)(25) = 400 - 400 = 0$. We will end up with ± 0 in the numerator. But it makes no difference if you add or subtract 0; you get the same answer. So this problem will have only one answer.

And finally, $3x^2 + 5x + 4$. Now, $b^2 - 4ac = 5^2 - 4(3)(4) = 25 - 48 = -23$. So in the numerator we will have $\sqrt{-23}$. Since you cannot take the square root of a negative number, there will be no solutions!

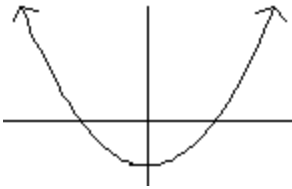
Summary: The Discriminant

- If the discriminant is a perfect square, you will have two rational solutions.
- If the discriminant is a positive number that is not a perfect square, you will have two irrational solutions (ie they will have square roots in them).
- If the discriminant is 0, you will have one solution.
- If the discriminant is negative, you will have no solutions.

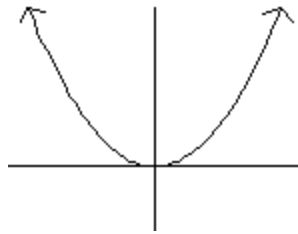
These rules do not have to be memorized: you can see them very quickly by understanding the quadratic formula (which **does** have to be memorized—if all else fails, try singing it to the tune of Frère Jacques).

Why is it that quadratic equations can have 2 solutions, 1 solution, or no solutions? This is easy to understand by looking at the following graphs. Remember that in each case the quadratic equation asks when the function is 0—that is to say, when it crosses the x -axis.

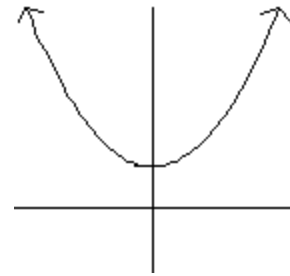
$$y = x^2 - 2 \text{ Equals } 0 \\ \text{two times}$$



$$y = x^2 \text{ Equals } 0 \\ \text{0 once}$$



$$y = x^2 + 2 \text{ . Never} \\ \text{equals } 0$$

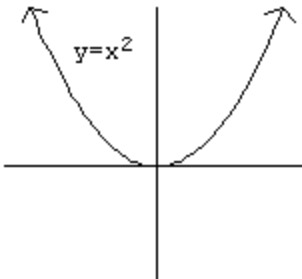


More on how to generate these graphs is given below. For the moment, the point is that you can visually see why a quadratic function can equal 0 twice, or one time, or never. It can **not** equal 0 three or more times.

Graphing Quadratic Equations

This module covers the graphing of quadratic equations.

The graph of the simplest quadratic function, $y = x^2$, looks like this:

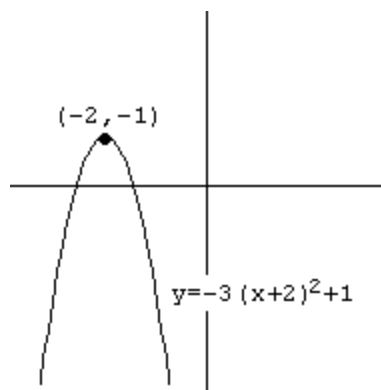


(You can confirm this by plotting points.) The point at the bottom of the U-shaped curve is known as the “vertex.”

Now consider the function $y = -3(x + 2)^2 + 1$. It’s an intimidating function, but we have all the tools we need to graph it, based on the **permutations** we learned in the first unit. Let’s step through them one by one.

- What does the $-$ sign do? It multiplies all y -values by -1 ; positive values become negative, and vice-versa. So we are going to get an **upside-down** U-shape. We say that $y = x^2$ “opens up” and $y = -x^2$ “opens down.”
- What does the 3 do? It multiplies all y -values by 3 ; positive values become more positive, and negative values become more negative. So it **vertical stretches** the function.
- What does the $+1$ at the end do? It adds 1 to all y -values, so it moves the function **up** by 1 .
- Finally, what does the $+2$ do? This is a horizontal modification: if we plug in $x = 10$, we will be evaluating the function at $x = 12$. In general, we will always be copying the original x^2 function to our **right**; so we will be 2 units to the **left** of it.

So what does the graph look like? It has moved 2 to the left and 1 up, so the vertex moves from the origin $(0,0)$ to the point $(-2,1)$. The graph has also flipped upside-down, and stretched out vertically.



So graphing quadratic functions is easy, no matter how complex they are, if you understand permutations—and if the functions are written in the form $y = a(x - h)^2 + k$, as that one was.

Graphing Quadratic Functions

The graph of a quadratic function is always a vertical parabola. If the function is written in the form $y = a(x - h)^2 + k$ then the vertex is at (h, k) . If a is positive, the parabola opens up; if a is negative, the parabola opens down.

But what if the functions are not expressed in that form? We're more used to seeing them written as $y = ax^2 + bx + c$. For such a function, you graph it by first putting it into the form we used above, and then graphing it. And the way you get it into the right form is...completing the square! This process is almost identical to the way we used completing the square to **solve** quadratic equations, but some of the details are different.

Example:

Graphing a Quadratic Function

Graphing a Quadratic Function

Graph $2x^2 - 20x + 58$

The problem.

$2(x^2 - 10x) + 58$

We used to start out by dividing both sides by the coefficient of x^2 (2 in this case). In this case, we don't have another side: we can't make that 2 go away. But it's still in the way of completing the square. So we factor it out of the first two terms. Do not factor it out of the third (numerical) term; leave that part alone, outside of the parentheses.

$2(x^2 - 10x + 25) + 58 - 50$

Inside the parentheses, add the number you need to complete the square. (Half of 10, squared.) Now, when we add 25 inside the parentheses, what we have really done to our function? We have added 50, since everything in parentheses is doubled. So we keep the function the same by subtracting that 50 right back again, outside the parentheses! Since all we have done in this step is add 50 and then subtract it, the function is unchanged.

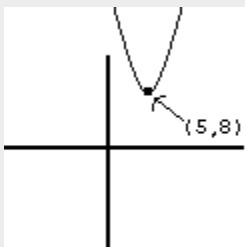
Graphing a Quadratic Function

$$2(x - 5)^2 + 8$$

Inside the parentheses, you now have a perfect square and can rewrite it as such. Outside the parentheses, you just have two numbers to combine.

Vertex (5,8) opens up

Since the function is now in the correct form, we can read this information straight from the formula and graph it. Note that the number inside the parentheses (the h) always changes sign; the number outside (the k) does not.



So there's the graph! It's easy to draw once you have the vertex and direction. It's also worth knowing that the 2 vertically stretches the graph, so it will be thinner than a normal x^2 .

This process may look intimidating at first. For the moment, don't worry about mastering the whole thing—instead, look over every **individual step** carefully and make sure you understand why it works—that is, why it keeps the function fundamentally unchanged, while moving us toward our goal of a form that we can graph.

The good news is, this process is basically the same every time. A different example is worked through in the worksheet “Graphing Quadratic

Functions II”—that example differs only because the x^2 term does not have a coefficient, which changes a few of the steps in a minor way. You will have plenty of opportunity to practice this process, which will help you get the “big picture” if you understand all the individual steps.

And don't forget that what we're really creating here is an algebraic generalization!

Equation:

$$2x^2 - 20x + 58 = 2(x - 5)^2 + 8$$

This is exactly the sort of generalization we discussed in the first unit—the assertion that these two very different functions will **always give the same answer** for any x -value you plug into them. For this very reason, we can also assert that the two graphs will look the same. So we can graph the first function by graphing the second.

Quadratic Concepts -- Solving Problems by Graphing Quadratic Functions

Surprisingly, there is a fairly substantial class of real world problems that can be solved by graphing quadratic functions.

These problems are commonly known as “optimization problems” because they involve the question: “When does this important function reach its **maximum**?” (Or sometimes, its **minimum**?) In real life, of course, there are many things we want to maximize—a company wants to maximize its revenue, a baseball player his batting average, a car designer the leg room in front of the driver. And there are many things we want to minimize—a company wants to minimize its costs, a baseball player his errors, a car designer the amount of gas used. Mathematically, this is done by writing a function for that quantity and finding where that function reaches its highest or lowest point.

Example:

Exercise:

Problem:

If a company manufactures x items, its total cost to produce these items is $x^3 - 10x^2 + 43x$. How many items should the company make in order to **minimize its average cost per item**?

Solution:

“Average cost per item” is the total cost, divided by the number of items. For instance, if it costs \$600 to manufacture 50 items, then the average cost per item was \$12. It is important for companies to minimize average cost because this enables them to sell at a low price.

In this case, the total cost is $x^3 - 10x^2 + 43x$ and the number of items is x . So the **average cost per item** is

Equation:

$$A(x) = \frac{x^3 - 10x^2 + 43x}{x} = x^2 - 10x + 43$$

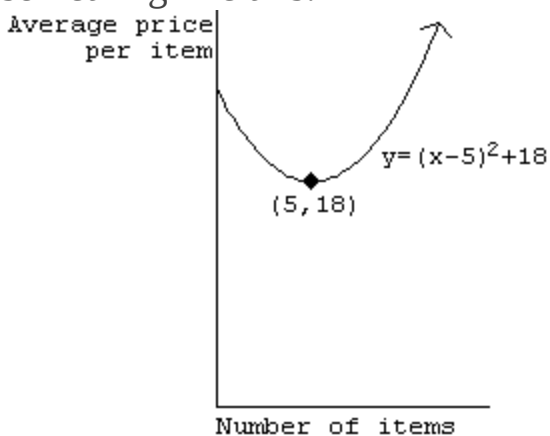
What the question is asking, mathematically, is: what value of x makes this function the **lowest**?

Well, suppose we were to graph this function. We would complete the square by rewriting it as:

Equation:

$$A(x) = x^2 - 10x + 25 + 18 = (x - 5)^2 + 18$$

The graph opens up (since the $(x - 5)^2$ term is positive), and has its vertex at $(5, 18)$ (since it is moved 5 to the right and 18 up). So it would look something like this:



I have graphed only the first quadrant, because negative values are not relevant for this problem (why?).

The real question here is, what can we learn from that graph? Every point on that graph represents one possibility for our company: if they manufacture x items, the graph shows what $A(x)$, the average cost per item, will be.

The point $(5,18)$ is the **lowest point on the graph**. It is possible for x to be higher or lower than 5, but it is never possible for A to be lower than 18. So if its goal is to minimize average cost, their best strategy is to manufacture 5 items, which will bring their average cost to \$18/item.

Quadratic Inequalities

This module covers quadratic inequalities.

Consider the following inequality:

Equation:

$$-x^2 + 6x - 8 \leq 0$$

There are a number of different ways to approach such a problem. The one that is stressed in the text is by **graphing**.

We begin by graphing the function $y = -x^2 + 6x - 8$. We know how to graph this function by completing the square, but we're going to take a shortcut—one that will not actually tell us what the vertex is, but that will give us what we need to know. The shortcut relies on finding only two facts about the quadratic function: what are its **roots** (the places it crosses the x -axis), and which way does it open?

Of course we know three ways to find the roots of a quadratic equation: the easiest is always factoring when it works, as it does in this case.

Equation:

$$-x^2 + 6x - 8 = 0$$

Equation:

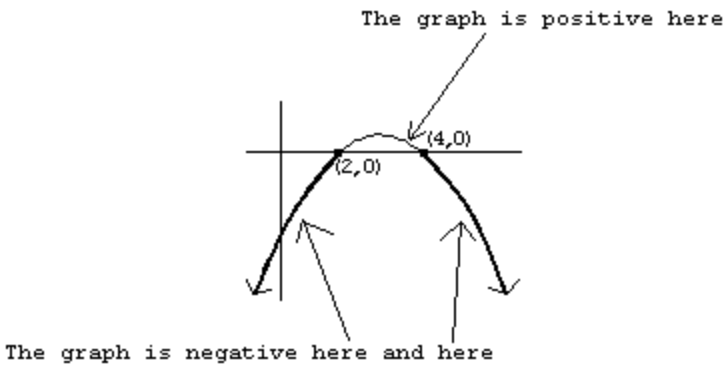
$$(x - 4)(-x + 2) = 0$$

Equation:

$$x = 0, x = 2$$

Second, which way does the parabola open? Since it is a vertical parabola with a **negative coefficient of the x^2 term**, it opens down.

So the graph looks like this:



(Hey, if it's that easy to graph a quadratic function, why did we spend all that time completing the square? Well, this method of graphing does not tell you the vertex. It tells us all we need to solve the quadratic inequality, but not everything about the graph. **Oh. Darn.** Anyway, back to our original problem.)

If that is the graph of $y = -x^2 + 6x - 8$, then let us return to the original question: when is $-x^2 + 6x - 8$ less than or equal to 0? What this question is asking is: when does this graph dip **below** the x -axis? Looking at the graph, the answer is clear: the graph is below the x -axis, and therefore the function is negative, whenever $x \leq 2$ or $x \geq 4$.

When students graph these functions, they tend to get the right answer. Where students go wrong is by trying to take “shortcuts.” They find where the function equals zero, and then attempt to quickly find a solution based on that. To see where this goes wrong, consider the following:

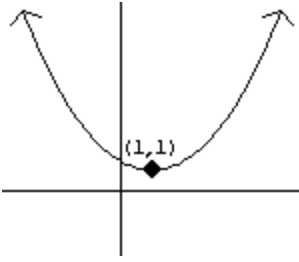
Equation:

$$x^2 - 2x + 2 \geq 0$$

If you attempt to find where this function **equals** zero—using, for instance, the quadratic formula—you will find that it never does. (Try it!) Many students will therefore quickly answer “no solution.” Quick, easy—and wrong. To see why, let's try graphing it.

Equation:

$$y = x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x - 1)^2 + 1$$



The original question asked: when is this function greater than or equal to zero? The graph makes it clear: **everywhere**. Any x value you plug into this function will yield a positive answer. So the solution is: “All real numbers.”

“Extra for Experts”: Quadratic Inequalities by Factoring

This technique is **not** stressed in this book. But it is possible to solve quadratic inequalities without graphing, if you can factor them.

Let’s return to the previous example:

Equation:

$$-x^2 + 6x - 8 \leq 0$$

Equation:

$$(x - 4)(-x + 2) \leq 0$$

When we solved quadratic **equations** by factoring, we asked the question: “How can you multiply two numbers and get 0?” (Answer: when one of them is 0.) Now we ask the question: “How can you multiply two numbers and get a **negative answer**?” Answer: when the two numbers have **different signs**. That is, the product will be negative if...

The first is positive and the	OR	The first is negative and
-------------------------------	----	---------------------------

second negative		the second positive
$x - 4 \geq 0$ AND $-x + 2 \leq 0$	OR	$x - 4 \leq 0$ AND $-x + 2 \geq 0$
$x \geq 4$ AND $-x \leq -2$	OR	$x \leq 4$ AND $-x \geq -2$
$x \geq 4$ AND $x \geq 2$	OR	$x \leq 4$ AND $x \leq 2$
$x \geq 4$	OR	$x \leq 2$

This is, of course, the same answer we got by graphing.

Exponent Concepts -- Introduction

An exponent means repeated multiplication. For instance, 10^6 means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, or 1,000,000.

You've probably noticed that there is a logical progression of operations. When you **add the same number repeatedly**, that's multiplication. When you **multiply the same number repeatedly**, that's an exponent.

However, there is one vital difference: addition and multiplication **commute**, but exponentiation does **not commute**. This is a fancy way of saying that order matters. $2 + 3 = 3 + 2$; and $2 \cdot 3 = 3 \cdot 2$, but 2^3 is **not** the same as 3^2 .

Laws of Exponents

This module covers the laws that govern the manipulation of exponents in algebra.

The following are generally referred to as the “laws” or “rules” of exponents.

Equation:

$$x^a x^b = x^{a+b}$$

Equation:

$$\frac{x^a}{x^b} = x^{a-b} \quad \text{or} \quad \frac{1}{x^{b-a}}$$

Equation:

$$(x^a)^b = x^{ab}$$

As with any formula, the most important thing is to be able to **use** them—that is, to understand what they mean. But it is also important to know **where these formulae come from**. And finally, in this case, the three should be memorized.

So...what do they mean? They are, of course, algebraic generalizations—statements that are true for any x , a , and b values. For instance, the first rule tells us that:

Equation:

$$7^{12} \cdot 7^4 = 7^{16}$$

which you can confirm on your calculator. Similarly, the third rule promises us that

Equation:

$$(7^{12})^4 = 7^{48}$$

These rules can be used to combine and simplify expressions.

Example:

Simplifying with the Rules of Exponents

$$\frac{(x^3)^4 \cdot x^5}{x^9 \cdot x^{11}}$$

$$= \frac{x^{12} \cdot x^5}{x^9 \cdot x^{11}}$$

Third rule: $(x^a)^b = x^{ab}$, where $a = 3$ and $b = 4$

$$= \frac{x^{17}}{x^{20}}$$

First rule: $x^a x^b = x^{a+b}$, done on both the top and bottom

$$= \frac{1}{x^3}$$

Second rule: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$, where we choose this form to avoid a negative exponent

Why do these rules work? It's very easy to see, based on what an exponent is.

Why does the first rule work?

Why does the first rule work?	
19^3	19^4
$= (19 \cdot 19 \cdot 19)$	$(19 \cdot 19 \cdot 19 \cdot 19)$
$= 19^7$	

You see what happened? 19^3 means three 19s multiplied; 19^4 means four 19s multiplied. Multiply them together, and you get **seven** 19s multiplied.

Why does the second rule work?	
First form	Second form
$\frac{19^8}{19^5}$	$\frac{19^5}{19^8}$
$= \frac{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}$	$= \frac{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}$
$= \frac{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot 19 \cdot 19 \cdot 19}{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19}}$	$= \frac{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19}}{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot 19 \cdot 19 \cdot 19}$
$= 19^3$	$= \frac{1}{19^3}$

In this case, the key is **fraction cancellations**. When the top is multiplied by 19 and the bottom is multiplied by 19, canceling these 19s has the effect of dividing **the top and bottom** by 19. When you divide the top and bottom of a fraction by the same number, the fraction is unchanged.

You can also think of this rule as the inevitable consequence of the **first** rule. If $19^3 \cdot 19^5 = 19^8$, then $\frac{19^8}{19^5}$ (which asks the question “ 19^5 times what equals 19^8 ?”) must be 19^3 .

Why does the third rule work?

$$(19^3)^4$$

$$= (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19)$$

$$= 19^{12}$$

What does it mean to raise something to the fourth power? It means to multiply it by itself, four times. In this case, what we are multiplying by itself four times is 19^3 , or $(19 \cdot 19 \cdot 19)$. Three 19s multiplied **four times** makes twelve 19s multiplied.

Zero, Negative Numbers, and Fractions as Exponents

This module covers how zero, negative numbers, and fractions can be used as exponents.

The definition of exponents given at the beginning of this section— 10^6 means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ —does **not** enable us to answer questions such as:

Equation:

$$4^0 = ?$$

Equation:

$$5^{-4} = ?$$

Equation:

$$9^{\frac{1}{2}} = ?$$

You can't "multiply 9 by itself half a time" or "multiply 5 by itself -4 times." In general, the original definition only applies if the exponent is a **positive integer**.

It's very important to understand this point. The question is **not** "What answer does our original definition give in these cases?" The original definition does **not** give any answer in these cases. If we want to include these numbers, we need a whole new definition of what an exponent is.

In principle, many such definitions are possible. We **could** define 5^{-4} as $5/5/5/5$: in other words, divide four times instead of multiplying four times. Or we **could** define 5^{-4} as 5^{-4} : take 5 to the fourth power, and then multiply it by -1 . Or we **could** define 5^{-4} as $-(5)^4$: take -5 to the fourth power (which gives a different answer from the previous definition). It seems at first that we are at liberty to choose any definition we want.

Given that degree of freedom, you may be very surprised at the definitions that are actually used: they seem far more arbitrary and complicated than some others you could come up with.

Zero exponents	Negative exponents	Fractional exponents (numerator = 1)	Fractional exponents (numerator \neq 1)
Always 1	Go in the denominator	Act as roots	The numerator is an exponent The denominator is a root
$7^0 = 1$ $9^0 = 1$ $x^0 = 1$	$7^{-3} = \frac{1}{7^3} = \frac{1}{343}$ $x^{-5} = \frac{1}{x^5}$ $\frac{1}{5^{-3}} = 5^3$	$9^{\frac{1}{2}} = \sqrt{9} = 3$ $2^{\frac{1}{2}} = \sqrt{2}$ $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ $x^{\frac{1}{4}} = \sqrt[4]{x}$	$8^{\frac{2}{3}} = \sqrt[3]{8^2}$ or $(\sqrt[3]{8})^2$ Order doesn't matter! $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $(\sqrt[3]{8})^2 = 2^2 = 4$ $8^{\frac{3}{2}} = \sqrt{8^3}$ or $(\sqrt{8})^3$

Definitions: When the exponent is not a positive integer

Note that you can **combine** these definitions. For instance, $8^{-\frac{2}{3}}$ is a negative, fractional exponent. The negative exponent means, as always, “put me in the denominator.” So we can write:

Equation:

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{4}$$

OK, so why define exponents that way?

These are obviously not chosen to be the **simplest** possible definitions. But they are chosen to be **consistent** with the behavior of positive-integer exponents.

One way to see that consistency is to consider the following progression:

Equation:

$$19^4 = 19 \cdot 19 \cdot 19 \cdot 19$$

Equation:

$$19^3 = 19 \cdot 19 \cdot 19$$

Equation:

$$19^2 = 19 \cdot 19$$

Equation:

$$19^1 = 19$$

What happens each time we decrease the exponent by 1? Your first response might be “we have one less 19.” But what is really happening, mathematically, to the numbers on the right? The answer is that, with each step, they are **dividing by 19**. If you take $19 \cdot 19 \cdot 19 \cdot 19$, and **divide it by 19**, you get $19 \cdot 19 \cdot 19$. Divide that by 19 again, and you get $19 \cdot 19$...and so on. From this we can formulate the following principle for the powers of 19:

Whenever you **subtract 1** from the exponent, you **divide the answer by 19**.

As I said earlier, we want the behavior of our new exponents to be consistent with the behavior of the **old** (positive-integer) exponents. So we can continue this progression as follows:

Equation:

$$19^0 = \frac{19}{19} = 1$$

Equation:

$$19^{-1} = \frac{1}{19}$$

Equation:

$$19^{-2} = \frac{\frac{1}{19}}{19} = \frac{1}{19^2}$$

...and so on. We can arrive at our definitions anything⁰ = 1 and **negative exponents go in the denominator** by simply requiring this progression to be consistent.

More rigorously, we can find all our exponent definitions by using the **laws of exponents**. For instance, what is 4⁰? We can approach this question indirectly by asking: what is $\frac{4^2}{4^2}$?

- The second law of exponents tells us that $\frac{4^2}{4^2} = 4^{2-2}$, which is of course 4⁰.
- But of course, $\frac{4^2}{4^2}$ is just $\frac{16}{16}$, or 1.
- Since $\frac{4^2}{4^2}$ is both 4⁰ **and** 1, 4⁰ and 1 must be the same thing!

The proofs given below all follow this pattern. They use the laws of exponents to rewrite expressions such as $\frac{4^2}{4^2}$, and go on to show how zero, negative, and fractional exponents must be defined. We started with the definition of an exponent for a positive integer, 10⁶ = 10 · 10 · 10 · 10 · 10 · 10. From there, we developed the laws of exponents. Now we find that, if we want those same laws to apply to other kinds of exponents, there is only one correct way to define those other kinds of exponents.

Zero exponents	Negative exponents	Fractional exponents (numerator = 1)	Fractional exponents (numerator ≠ 1)
Always 1	Go in the denominator	Act as roots	The numerator is an exponent The denominator is a root
$\frac{4^2}{4^2} = 4^{2-2} = 4^0$ but $\frac{4^2}{4^2} = \frac{16}{16} = 1$ so 4 ⁰ must be 1!	$\frac{10^1}{10^3} = 10^{1-3} = 10^{-2}$ but $\frac{10^1}{10^3} = \frac{10}{10 \cdot 10 \cdot 10} = \frac{1}{10 \cdot 10}$ so 10 ⁻² must be $\frac{1}{10^2}$	$(9^{\frac{1}{2}})^2 = 9^{\frac{1}{2} \cdot 2} = 9^1 = 9$ So what is 9 ^{1/2} ? Well, when you square it, you get 9. So it must be $\sqrt{9}$, or 3!	$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2}$

Proofs: When the exponent is not a positive integer

You may want to experiment with making these proofs more general and more rigorous by using letters instead of numbers. For instance, in the third case, we could write:

Equation:

$$(x^{\frac{1}{a}})^a = x^{(\frac{1}{a})(a)} = x^1$$

Equation:

$$(x^{\frac{1}{a}})^a = x$$

Equation:

$$\sqrt[a]{(x)^a} = \sqrt[a]{x}$$

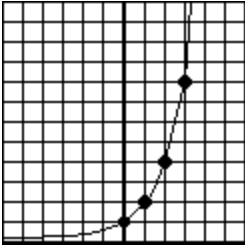
Equation:

$$x^{\frac{1}{2}} = \sqrt{x}$$

Exponential Curves

This module discusses the graphing of exponential curves.

By plotting points, you can discover that the graph of $y = 2^x$ looks like this:

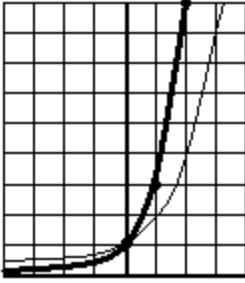


$$y = 2^x$$

A few points to notice about this graph.

- It goes through the point $(0,1)$ because $2^0 = 1$.
- It never dips below the x -axis. The **domain** is unlimited, but the **range** is $y > 0$. (*Think about our definitions of exponents: whether x is positive or negative, integer or fraction, 2^x is **always** positive.)
- Every time you move one unit to the right, the graph height doubles. For instance, 2^5 is twice 2^4 , because it multiplies by **one more** 2. So as you move to the right, the y -values start looking like 8, 16, 32, 64, 128, and so on, going up more and more sharply.
- Conversely, every time you move one unit to the left, the graph height drops in half. So as you move to the left, the y -values start looking like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and so on, falling closer and closer to 0.

What would the graph of $y = 3^x$ look like? Of course, it would also go through $(0,1)$ because $3^0 = 1$. With each step to the right, it would **triple**; with each step to the left, it would drop in a **third**. So the overall shape would look similar, but the rise (on the right) and the drop (on the left) would be faster.



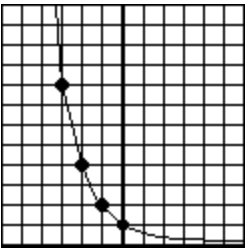
$y = 2^x$ in
thin line;

$y = 3^x$ in
thick line;

They
cross at
(0,1)

As you might guess, graphs such as 5^x and 10^x all have this same characteristic shape. In fact, any graph a^x where $a > 1$ will look basically the same: starting at $(0,1)$ it will rise more and more sharply on the right, and drop toward zero on the left. This type of graph models **exponential growth**—functions that keep multiplying by the same number. A common example, which you work through in the text, is compound interest from a bank.

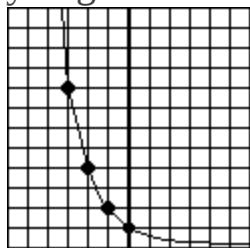
The opposite graph is $\left(\frac{1}{2}\right)^x$.



$$y = \left(\frac{1}{2}\right)^x$$

Each time you move to the right on this graph, it multiplies by $\frac{1}{2}$: in other words, it **divides** by 2, heading closer to zero the further you go. This kind of equation is used to model functions that keep **dividing** by the same number; for instance, radioactive decay. You will also be working through examples like this one.

Of course, all the **permutations** from the first chapter on “functions” apply to these graphs just as they apply to any graph. A particularly interesting example is 2^{-x} . Remember that when you replace x with $-x$, $f(3)$ becomes the old $f(-3)$ and vice-versa; in other words, the graph flips around the y -axis. If you take the graph of 2^x and permute it in this way, you get a familiar shape:



$$y = 2^{-x}$$

Yes, it's $\left(\frac{1}{2}\right)^x$ in a new disguise!

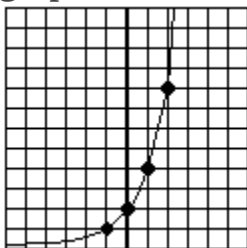
Why did it happen that way? Consider that $\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x}$. But 1^x is just 1 (in other words, 1 to the **anything** is 1), so $\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$. But negative exponents go in the denominator: $\frac{1}{2^x}$ is the same thing as 2^{-x} ! So we arrive at: $\left(\frac{1}{2}\right)^x = 2^{-x}$. The two functions are the same, so their graphs are of course the same.

Another fun pair of permutations is:

$y = 2 \cdot 2^x$ **Looks just like $y = 2^x$ but vertically stretched: all y-values double**

$y = 2^{x+1}$ Looks just like $y = 2^x$ but horizontally shifted: moves 1 to the left

If you permute 2^x in these two ways, you will find that they create the same graph.



$$y = 2 \cdot 2^x$$

aka

$$y = 2^{x+1}$$

Once again, this is predictable from the rules of exponents:

$$2 \cdot 2^x = 2^1 \cdot 2^x = 2^{x+1}$$

Using exponential functions to model behavior

In the first chapter, we talked about **linear functions** as functions that **add the same amount every time**. For instance, $y = 3x + 4$ models a function that starts at 4; every time you increase x by 1, you add 3 to y .

Exponential functions are conceptually very analogous: they **multiply by the same amount every time**. For instance, $y = 4 \times 3^x$ models a function that starts at 4; every time you increase x by 1, you **multiply** y by 3.

Linear functions can go down, as well as up, by having **negative slopes**: $y = -3x + 4$ starts at 4 and **subtracts** 3 every time. Exponential functions can go down, as well as up, by having **fractional bases**: $y = 4 \times (\frac{1}{3})^x$ starts at 4 and **divides** by 3 every time.

Exponential functions often defy intuition, because they grow much faster than people expect.

Modeling exponential functions

Your father's house was worth \$100,000 when he bought it in 1981. Assuming that it increases in value by 8% every year, what was the house worth in the year 2001? (*Before you work through the math, you may want to make an intuitive guess as to what you think the house is worth. Then, after we crunch the numbers, you can check to see how close you got.)

Often, the best way to approach this kind of problem is to begin by making a chart, to get a sense of the growth pattern.

Year	Increase in Value	Value
1981	N/A	100,000
1982	8% of 100,000 = 8,000	108,000
1983	8% of 108,000 = 8,640	116,640
1984	8% of 116,640 = 9,331	125,971

Before you go farther, make sure you understand where the numbers on that chart come from. It's OK to use a calculator. But if you blindly follow the numbers without understanding the calculations, the whole rest of this section will be lost on you.

In order to find the pattern, look at the "Value" column and ask: what is happening to these numbers every time? Of course, we are adding 8% each time, but what does that really mean? With a little thought—or by looking

at the numbers—you should be able to convince yourself that the numbers are **multiplying by 1.08 each time**. That’s why this is an exponential function: the value of the house multiplies by 1.08 every year.

So let’s make that chart again, in light of this new insight. Note that I can now skip the middle column and go straight to the answer we want. More importantly, note that I am not going to use my calculator this time—I don’t want to multiply all those 1.08s, I just want to note each time that the answer is **1.08 times the previous answer**.

Year	House Value
1981	100,000
1982	$100,000 \times 1.08$
1983	$100,000 \times 1.08^2$
1984	$100,000 \times 1.08^3$
1985	$100,000 \times 1.08^4$
y	$100,000 \times 1.08^{\text{something}}$

If you are not clear where those numbers came from, think again about the conclusion we reached earlier: each year, the value **multiplies by 1.08**. So if the house is worth $100,000 \times 1.08^2$ in 1983, then its value in 1984 is $(100,000 \times 1.08^2) \times 1.08$, which is $100,000 \times 1.08^3$.

Once we write it this way, the pattern is clear. I have expressed that pattern by adding the last row, the value of the house in any year y . And what is the

mystery exponent? We see that the exponent is 1 in 1982, 2 in 1983, 3 in 1984, and so on. In the year y , the exponent is $y - 1981$.

So we have our house value function:

Equation:

$$v(y) = 100,000 \times 1.08^{y-1981}$$

That is the pattern we needed in order to answer the question. So in the year 2001, the value of the house is $100,000 \times 1.08^{20}$. Bringing the calculator back, we find that the value of the house is now \$466,095 and change.

Wow! The house is over four times its original value! That's what I mean about exponential functions growing faster than you expect: they start out slow, but given time, they explode. This is also a practical life lesson about the importance of saving money early in life—a lesson that many people don't realize until it's too late.

Logarithm Concepts

This module introduces the concept of logarithms.

Suppose you are a biologist investigating a population that doubles every year. So if you start with 1 specimen, the population can be expressed as an exponential function: $p(t) = 2^t$ where t is the number of years you have been watching, and p is the population.

Question: How long will it take for the population to exceed 1,000 specimens?

We can rephrase this question as: “2 to **what power** is 1,000?” This kind of question, where you know the base and are looking for the exponent, is called a **logarithm**.

$\log_2 1000$ (read, “the logarithm, base two, of a thousand”) means “2, raised to what power, is 1000?”

In other words, the logarithm always asks “**What exponent should we use?**” This unit will be an exploration of logarithms.

A few quick examples to start things off

Problem	Means	The answer is	because
$\log_2 8$	2 to what power is 8?	3	2^3 is 8
$\log_2 16$	2 to what power is	4	2^4 is 16

	16?		
$\log_2 10$	2 to what power is 10?	somewhere between 3 and 4	$2^3 = 8$ and $2^4 = 16$
$\log_8 2$	8 to what power is 2?	$\frac{1}{3}$	$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
$\log_{10} 10,000$	10 to what power is 10,000?	4	$10^4 = 10,000$
$\log_{10} \left(\frac{1}{100}\right)$	10 to what power is $\frac{1}{100}$?	-2	$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$
$\log_5 0$	5 to what power is 0?	There is no answer	$5^{\text{something}}$ will never be 0

As you can see, one of the most important parts of finding logarithms is being very familiar with how exponents work!

Logarithm Concepts -- The logarithm explained by analogy to roots

The logarithm may be the first really new concept you've encountered in Algebra II. So one of the easiest ways to understand it is by comparison with a familiar concept: roots.

Suppose someone asked you: "Exactly what does root mean?" You do understand roots, but they are difficult to define. After a few moments, you might come up with a definition very similar to the "question" definition of logarithms given above. $\sqrt[3]{8}$ means "what number cubed is 8?"

Now the person asks: "How do you find roots?" Well...you just play around with numbers until you find one that works. If someone asks for $\sqrt{25}$, you just have to know that $5^2 = 25$. If someone asks for $\sqrt{30}$, you know that has to be bigger than 5 and smaller than 6; if you need more accuracy, it's time for a calculator.

All that information about roots applies in a very analogous way to logarithms.

	Roots	Logs
The question	$\sqrt[n]{x}$ means "what number, raised to the a power, is x?" As an equation, $?^a = x$	$\log_a x$ means "a, raised to what power, is x?" As an equation, $a^? = x$
Example that comes out even	$\sqrt[3]{8} = 2$	$\log_2 8 = 3$

	Roots	Logs
Example that doesn't	$\sqrt[3]{10}$ is a bit more than 2	$\log_2 10$ is a bit more than 3
Out of domain example	$\sqrt{-4}$ does not exist (x^2 will never give -4)	$\log_2(0)$ and $\log_2(-1)$ do not exist (2^x will never give 0 or a negative answer)

Rewriting Logarithm Equations as Exponent Equations

This module discusses how equations involving logarithms can be rewritten using exponents.

Both root equations and logarithm equations can be rewritten as exponent equations.

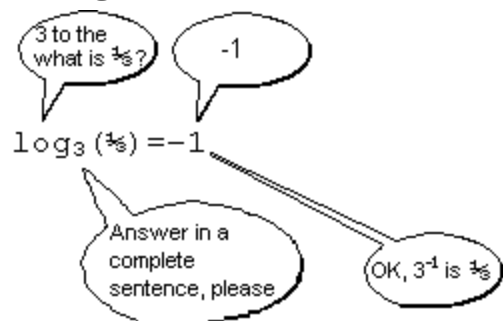
$\sqrt{9} = 3$ can be rewritten as $3^2 = 9$. These two equations are the same statement about numbers, written in two different ways. $\sqrt{9}$ asks the question “What number squared is 9?” So the equation $\sqrt{9} = 3$ asks this question, and then answers it: “3 squared is 9.”

We can rewrite logarithm equations in a similar way. Consider this equation:

Equation:

$$\log_3 \frac{1}{3} = -1$$

If you are asked to rewrite that logarithm equation as an exponent equation, think about it this way. The left side asks: “3 to what power is $\frac{1}{3}$?” And the right side answers: “3 to the **-1 power** is $\frac{1}{3}$.” $3^{-1} = \frac{1}{3}$.



These two equations, $\log_3 \frac{1}{3} = -1$ and $3^{-1} = \frac{1}{3}$, are two different ways of expressing the same numerical relationship.

The Logarithm Defined as an Inverse Function

This module discusses how logarithms can be defined as inverse functions.

\sqrt{x} can be defined as the **inverse function** of x^2 . Recall the definition of an inverse function— $f^{-1}(x)$ is defined as the inverse of $f^1(x)$ if it reverses the inputs and outputs. So we can demonstrate this inverse relationship as follows:

\sqrt{x} is the inverse function of x^2
$3 \rightarrow x^2 \rightarrow 9$
$9 \rightarrow \sqrt{x} \rightarrow 3$

Similarly, $\log_2 x$ is the **inverse function** of the exponential function 2^x .

$\log_2 x$ is the inverse function of 2^x
$3 \rightarrow 2^x \rightarrow 8$
$8 \rightarrow \log_2 x \rightarrow 3$

(You may recall that during the discussion of inverse functions, 2^x was the only function you were given that you **could not find** the inverse of. Now you know!)

In fact, as we noted in the first chapter, \sqrt{x} is **not** a perfect inverse of x^2 , since it does not work for negative numbers. $(-3)^2 = 9$, but $\sqrt{9}$ is not -3 . Logarithms have no such limitation: $\log_2 x$ is a perfect inverse for 2^x .

The inverse of addition is subtraction. The inverse of multiplication is division. Why do exponents have two completely different kinds of inverses, roots and logarithms? **Because exponents do not commute.** 3^2 and 2^3 are not the same number. So the question “what number squared equals 10?” and the question “2 to what power equals 10?” are different questions, which we express as $\sqrt{10}$ and $\log_2 10$, respectively, and they have different answers. x^2 and 2^x are not the same function, and they therefore have different inverse functions \sqrt{x} and $\log_2 10$.

Properties of Logarithms

This module contains some of the properties of logarithms and how they can be used for manipulation.

Just as there are three fundamental laws of exponents, there are three fundamental laws of logarithms.

Equation:

$$\log_x(ab) = \log_x a + \log_x b$$

Equation:

$$\log_x \frac{a}{b} = \log_x a - \log_x b$$

Equation:

$$\log_x(a^b) = b\log_x a$$

As always, these **algebraic generalizations** hold for any a , b , and x .

Example:

Properties of Logarithms

1. Suppose you are given these two facts:

- $\log_4 5 = 1.16$
- $\log_4 10 = 1.66$

2. Then we can use the laws of logarithms to conclude that:

- $\log_4(50) = \log_4 5 + \log_4 10 = 2.82$
- $\log_4(2) = \log_4 10 - \log_4 5 = 0.5$
- $\log_4(100,000) = 5\log_4 10 = 8.3$

Note:All three of these results can be found quickly, and without a calculator. Note that the second result could also be figured out directly, since $4^{\frac{1}{2}} = 2$.

These properties of logarithms were very important historically, because they enabled pre-calculator mathematicians to perform **multiplication** (which is very time-consuming and error prone) by doing **addition** (which is faster and easier). These rules are still useful in simplifying complicated expressions and solving equations.

Example:

Solving an equation with the properties of logarithms

$\log_2 x - \log_2(x - 1) = 5$	The problem
$\log_2\left(\frac{x}{x-1}\right) = 5$	Second property of logarithms
$\frac{x}{x-1} = 2^5 = 32$	Rewrite the log as an exponent. (2-to-what is? $\frac{x}{x-1}$ 2-to-the-5!)
$x = 32(x - 1)$	Multiply. We now have an easy equation to solve.
$x = 32x - 32$	

$-31x = -32$	
$x = \frac{32}{31}$	

Proving the Properties of Logarithms

If you understand what an exponent is, you can very quickly see why the three rules of exponents work. But why do logarithms have these three properties?

As you work through the text, you will demonstrate these rules intuitively, by viewing the logarithm as a **counter**. ($\log_2 8$ asks “**how many** 2s do I need to multiply, in order to get 8?”) However, these rules can also be rigorously proven, using the laws of exponents as our starting place.

Proving the First Law of Logarithms, $\log_x(ab) = \log_x a + \log_x b$	
$m = \log_x a$	I’m just inventing m to represent this log
$x^m = a$	Rewriting the above expression as an exponent. ($\log_x a$ asks “x to what power is a ?” And the equation answers: “x to the m is a .”)
$n = \log_x b$	Similarly, n will represent the other log.
$x^n = b$	

$\log_x(ab) = \log_x(x^m x^n)$	Replacing a and b based on the previous equations
$= \log_x(x^{m+n})$	This is the key step! It uses the first law of exponents. Thus you can see that the properties of logarithms come directly from the laws of exponents.
$= m + n$	$= \log_x(x^{m+n})$ asks the question: “ x to what power is x^{m+n} ?” Looked at this way, the answer is obviously $(m + n)$. Hence, you can see how the logarithm and exponential functions cancel each other out, as inverse functions must.
$= \log_x a + \log_x b$	Replacing m and n with what they were originally defined as. Hence, we have proven what we set out to prove.

To test your understanding, try proving the second law of logarithms: the proof is very similar to the first. For the third law, you need invent only one variable, $m = \log_x a$. In each case, you will rely on a different one of the three rules of exponents, showing how each exponent law corresponds to one of the logarithms laws.

Common Logarithms

This module covers some of the logarithms commonly encountered in algebra.

When you see a root without a number in it, it is assumed to be a **square** root. That is, $\sqrt{25}$ is a shorthand way of writing $\sqrt[2]{25}$. This rule is employed because **square** roots are more common than other types.

When you see a logarithm without a number in it, it is assumed to be a **base 10** logarithm. That is, $\log(1000)$ is a shorthand way of writing $\log_{10}(1000)$. A base 10 logarithm is also known as a “common” log.

Why are common logs particularly useful? Well, what is $\log_{10}(1000)$? By now you know that this asks the question “10 to what power is 1000?” The answer is 3. Similarly, you can confirm that:

Equation:

$$\log(10) = 1$$

Equation:

$$\log(100) = 2$$

Equation:

$$\log(1,000,000) = 6$$

We can also follow this pattern backward:

Equation:

$$\log(1) = 0$$

Equation:

$$\log\left(\frac{1}{10}\right) = -1$$

Equation:

$$\log\left(\frac{1}{100}\right) = -2$$

and so on. In other words, the common log tells you the **order of magnitude** of a number: how many zeros it has. Of course, $\log_{10}(500)$ is difficult to determine exactly without a calculator, but we can say immediately that it must be somewhere between 2 and 3, since 500 is between 100 and 1000.

Graphing Logarithmic Functions

This module discusses the graphing and plotting of logarithmic functions and some of their applications.

Suppose you want to graph the function $y = \log_2(x)$. You might start by making a table that looks something like this:

x	$y = \log_2(x)$
1	0
2	1
3	um...I'm not sure
4	2
5	can I use a calculator?

This doesn't seem to be the right strategy. Many of those numbers are just too hard to work with.

So, you start looking for numbers that **are** easy to work with. And you remember that it's important to look at numbers that are less than 1, as well as greater. And eventually, you end up with something more like this.

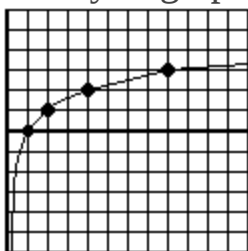
x	$y = \log_2(x)$
-----	-----------------

x	$y = \log_2(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

As long as you keep putting powers of 2 in the x column, the y column is very easy to figure.

In fact, the easiest way to generate this table is to recognize that it is the table of $y = 2^x$ values, only with the x and y coordinates switched! In other words, we have re-discovered what we already knew: that $y = 2^x$ and $y = \log_2(x)$ are inverse functions.

When you graph it, you end up with something like this:



$$y = \log_2(x)$$

As always, you can learn a great deal about the log function by reading the graph.

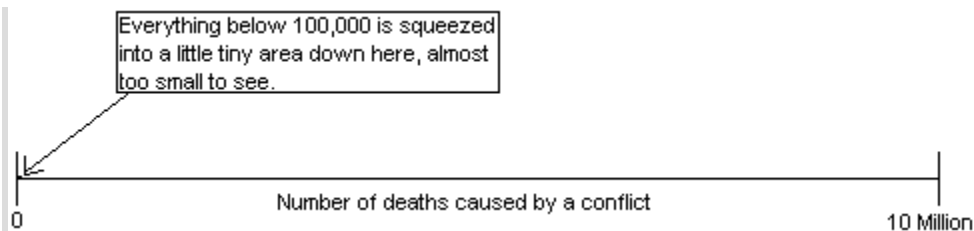
- The domain is $x > 0$. (You can't take the log of 0 or a negative number—do you remember why?).
- The range, on the other hand, is all numbers. Of course, all this inverts the function 2^x , which has a **domain** of all numbers and a **range** of $y > 0$.
- As x gets closer and closer to 0, the function dives down to smaller and smaller negative numbers. So the y -axis serves as an “asymptote” for the graph, meaning a line that the graph **approaches closer and closer to** without ever touching.
- As x moves to the right, the graph grows—but more and more slowly. As x goes from 4 to 8, the graph goes up by 1. As x goes from 8 to 16, the graph goes up by another 1. It doesn't make it up another 1 until x reaches 32...and so on.

This pattern of **slower and slower growth** is one of the most important characteristics of the log. It can be used to “slow down” functions that have too wide a range to be practical to work with.

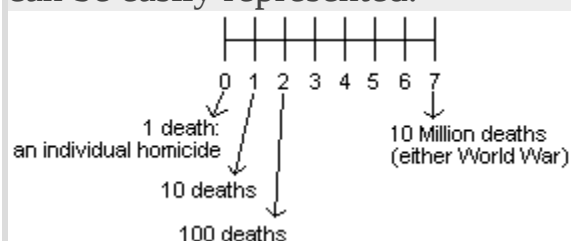
Example:

Using the log to model a real world problem

Lewis Fry Richardson (1881–1953) was a British meteorologist and mathematician. He was also an active Quaker and committed pacifist, and was one of the first men to apply statistics to the study of human conflict. Richardson catalogued 315 wars between 1820 and 1950, and categorized them by **how many deaths** they caused. At one end of the scale is a deadly quarrel, which might result in 1 or 2 deaths. At the other extreme are World War I and World War II, which are responsible for roughly 10 million deaths each.



As you can see from the chart above, working with these numbers is extremely difficult: on a scale from 0 to 10 Million, there is no visible difference between (say) 1 and 100,000. Richardson solved this problem by taking the **common log of the number of deaths**. So a conflict with 1,000 deaths is given a magnitude of $\log(1000) = 3$. On this scale, which is now the standard for conflict measurement, the magnitudes of all wars can be easily represented.



Richardson's scale makes it practical to chart, discuss, and compare wars and battles from the smallest to the biggest. For instance, he discovered that each time you move up by one on the scale—that is, each time the number of deaths multiplies by 10—the number of conflicts drops in a third. (So there are roughly three times as many “magnitude 5” wars as “magnitude 6,” and so on.)

The log is useful here because the logarithm function itself **grows so slowly** that it compresses the entire 1-to-10,000,000 range into a 0-to-7 scale. As you will see in the text, the same trick is used—for the same reason—in fields ranging from earthquakes to sound waves.

Rational Expression Concepts

This module provides an introduction to rational expressions.

The term “rational” in math is not used in the sense of “sane” or “sensible.” It is instead used to imply a **ratio**, or fraction. A **rational expression** is the ratio of two polynomials: for instance, $\frac{x}{x}$ is a rational expression.

There are two rules for working with rational expressions.

1. Begin every problem by factoring everything you can.
2. Remember that, despite all the complicated looking functions, a rational expression is just a fraction: you manipulate them using all the rules of fractions that you are familiar with.

Simplifying Rational Expressions

This module provides techniques for simplifying rational expressions.

How do you **simplify** a fraction? The answer is, you divide the top and bottom by the same thing.

Equation:

$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

So $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of writing the same number.

On the left, a pizza divided into six equal slices: the four shaded-in regions represent $\frac{4}{6}$ of a pizza. On the right, a pizza divided into three equal slices: the two shaded-in regions represent $\frac{2}{3}$ of a pizza. The two areas are identical: $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of expressing the **same amount of pizza**.

In some cases, you have to repeat this process more than once before the fraction is fully simplified.

Equation:

$$\frac{40}{48} = \frac{40 \div 4}{48 \div 4} = \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

It is vital to remember that **we have not divided this fraction by 4, or by 2, or by 8**. We have rewritten the fraction in another form: $\frac{40}{48}$ is the **same number** as $\frac{5}{6}$. In strictly practical terms, if you are given the choice

between $\frac{40}{48}$ of a pizza or $\frac{5}{6}$ of a pizza, it does not matter which one you choose, because they are the same amount of pizza.

You can divide the top and bottom of a fraction by the same number, but you cannot subtract the same number from the top and bottom of a fraction!

$$\frac{40}{48} = \frac{40-39}{48-39} = \frac{1}{9} \quad \times \text{ Wrong!}$$

Given the choice, a hungry person would be wise to choose $\frac{40}{48}$ of a pizza instead of $\frac{1}{9}$.

Dividing the top and bottom of a fraction by the same number leaves the fraction unchanged, and that is how you simplify fractions. **Subtracting** the same number from the top and bottom changes the value of the fraction, and is therefore an illegal simplification.

All this is review. But if you understand these basic fraction concepts, you are ahead of many Algebra II students! And if you can **apply these same concepts when variables are involved**, then you are ready to simplify rational expressions, because there are no new concepts involved.

As an example, consider the following:

Equation:

$$\frac{x^2 - 9}{x^2 + 6x + 9}$$

You might at first be tempted to cancel the common x^2 terms on the top and bottom. But this would be, mathematically, **subtracting** x^2 from both the top and the bottom; which, as we have seen, is an illegal fraction operation.



$\frac{x^2-9}{x^2+6x+9}$	$= \frac{\cancel{x} - 9}{\cancel{x} + 6x + 9}$	$= \frac{-9}{6x+9}$	✗ Wrong!
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To properly simplify this expression, begin by factoring both the top and the bottom, and **then** see if anything cancels.

**Example:
Simplifying Rational Expressions**

$\frac{x^2-9}{x^2+6x+9}$	The problem
$= \frac{(x+3)(x-3)}{(x+3)^2}$	Always begin rational expression problems by factoring! This factors easily, thanks to $(x + a)(x - a) = x^2 - a^2$ and $(x + a)^2 = x^2 + 2ax + a^2$
$= \frac{x-3}{x+3}$	Cancel a common $(x + 3)$ term on both the top and the bottom. This is legal because this term was multiplied on both top and bottom; so we are effectively dividing the top and bottom by $(x + 3)$, which leaves the fraction unchanged.

What we have created, of course, is an algebraic generalization:
Equation:

$$\frac{x^2 - 9}{x^2 + 6x + 9} = \frac{x - 3}{x + 3}$$

For any x value, the complicated expression on the left will give the same answer as the much simpler expression on the right. You may want to try one or two values, just to confirm that it works.

As you can see, the skills of **factoring** and **simplifying fractions** come together in this exercise. No new skills are required.

Multiplying Rational Expressions

This module covers the multiplication of rational expressions.

Multiplying fractions is easy: you just multiply the tops, and multiply the bottoms. For instance,

Equation:

$$\frac{6}{7} \times \frac{7}{11} = \frac{6 \times 7}{7 \times 11} = \frac{42}{77}$$

Now, you may notice that $\frac{42}{77}$ can be simplified, since 7 goes into the top and bottom. $\frac{42}{77} = \frac{42 \div 7}{77 \div 7} = \frac{6}{11}$. So $\frac{42}{77}$ is the correct answer, but $\frac{6}{11}$ is **also** the correct answer (since they are the same number), and it's a good bit simpler.

In fact, we could have jumped straight to the simplest answer first, and avoided dealing with all those big numbers, if we had noticed that we have a 7 in the numerator and a 7 in the denominator, and **cancelled** them before we even multiplied!

$$\frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{7}}{11} = \frac{6}{11}$$

This is a great time-saver, and you're also a lot less likely to make mistakes.

When multiplying fractions...

If the same number appears on the top and the bottom, you can cancel it before you multiply. This works regardless of whether the numbers appear in the **same fraction or different fractions**.

But it's critical to remember that this rule **only applies when you are multiplying fractions**: not when you are adding, subtracting, or dividing.

As you might guess, all this review of basic fractions is useful because, once again, rational expressions work the same way.

**Example:
Multiplying Rational Expressions**

$\frac{3x^2-21x-24}{x^2-16} \cdot \frac{x^2-6x+8}{3x+3}$	<p>The problem</p>
$= \frac{3(x-8)(x+1)}{(x+4)(x-4)} \cdot \frac{(x-2)(x-4)}{3(x+1)}$	<p>Always begin rational expression problems by factoring! Note that for the first element you begin by factoring out the common 3, and then factoring the remaining expression.</p>
$= \frac{\cancel{3}(x-8)\cancel{(x+1)}}{(x+4)\cancel{(x-4)}} \cdot \frac{(x-2)\cancel{(x-4)}}{\cancel{3}(x+1)}$	<p>When multiplying fractions, you can cancel anything on top with anything on the bottom, even across different fractions</p>
$= \frac{(x-8)(x-2)}{x+4}$	<p>Now, just see what you're left with. Note that you could rewrite the top as $x^2 - 10x + 16$ but it's generally easier to work with in factored form.</p>

Dividing Rational Expressions

To divide fractions, you flip the bottom one, and then multiply.

Equation:

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

After the “flipping” stage, all the considerations are exactly the same as multiplying.

Example:
Dividing Rational Expressions

$\frac{\frac{x^2-3x}{2x^2-13x+6}}{\frac{x^3+4x}{x^2-12x+36}}$	<p>This problem could also be written as: $\frac{x^2-3x}{2x^2-13x+6} \div \frac{x^3+4x}{x^2-12x+36}$. However, the \div symbol is rarely seen at this level of math. $12 \div 4$ is written as $\frac{12}{4}$.</p>
$\frac{x^2-3x}{2x^2-13x+6} \times \frac{x^2-12x+36}{x^3+4x}$	<p>Flip the bottom and multiply. From here, it's a straight multiplication problem.</p>
$= \frac{x(x-3)}{(2x-1)(x-6)} \times \frac{(x-6)^2}{x(x^2+4)}$	<p>Always begin rational expression problems by factoring! Now, cancel a factor of x and an $(x-6)$ and you get...</p>
$= \frac{(x-3)(x-6)}{(2x-1)(x^2+4)}$	<p>That's as simple as it gets, I'm afraid. But it's better than what we started with!</p>

Adding and Subtracting Rational Expressions

This module covers the addition and subtraction of rational expressions.

Adding and subtracting fractions is harder—but once again, it is a familiar process.

Equation:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

The key is finding the **least common denominator**: the smallest multiple of both denominators. Then you rewrite the two fractions with this denominator. Finally, you add the fractions by **adding the numerators and leaving the denominator alone**.

But how do you find the least common denominator? Consider this problem:

Equation:

$$\frac{5}{12} + \frac{7}{30}$$

You could probably find the least common denominator if you played around with the numbers long enough. But what I want to show you is a **systematic method** for finding least common denominators—a method that works with rational expressions just as well as it does with numbers. We start, as usual, by factoring. For each of the denominators, we find all the **prime factors**, the prime numbers that multiply to give that number.

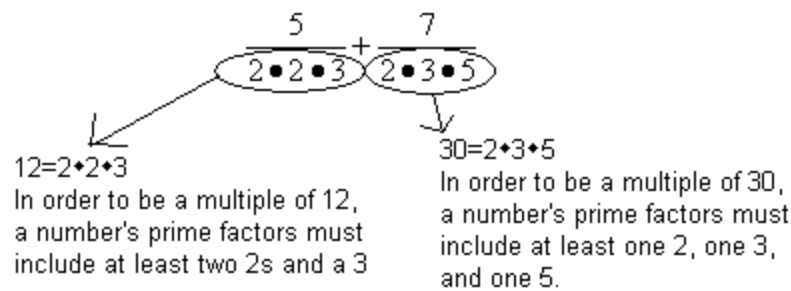
Equation:

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5}$$

If you are not familiar with the concept of prime factors, it may take a few minutes to get used to. $2 \times 2 \times 3$ is 12, broken into its **prime factors**: that

is, it is the list of prime numbers that multiply to give 12. Similarly, the prime factors of 30 are $2 \times 3 \times 5$.

Why does that help? Because $12 = 2 \times 2 \times 3$, any number whose prime factors include two 2s and one 3 will be a multiple of 12. Similarly, any number whose prime factors include a 2, a 3, and a 5 will be a multiple of 30.



The least common denominator is the smallest number that meets both these criteria: it must have two 2s, one 3, and one 5. Hence, the least common denominator must be $2 \times 2 \times 3 \times 5$, and we can finish the problem like this.

Equation:

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5} = \frac{5\underline{5}}{(2 \cdot 2 \cdot 3)\underline{5}} + \frac{7\underline{2}}{(2 \cdot 3 \cdot 5)\underline{2}} = \frac{25}{60} + \frac{14}{60} = \frac{39}{60}$$

This may look like a very strange way of solving problems that you've known how to solve since the third grade. However, I would urge you to spend a few minutes carefully following that solution, focusing on the question: **why is $2 \times 2 \times 3 \times 5$ guaranteed to be the least common denominator?** Because once you understand that, you have the key concept required to add and subtract rational expressions.

Example:
Subtracting Rational Expressions

$\frac{3}{x^2+12x+36} - \frac{4x}{x^3+4x^2-12x}$	<p>The problem</p>
$= \frac{3}{(x+6)^2} - \frac{4x}{x(x+6)(x-2)}$	<p>Always begin rational expression problems by factoring! The least common denominator must have two $(x + 6)$ s, one x , and one $(x - 2)$.</p>
$= \frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} - \frac{4x(x+6)}{x(x+6)^2(x-2)}$	<p>Rewrite both fractions with the common denominator.</p>
$= \frac{3(x)(x-2)-4x(x+6)}{x(x-2)(x+6)^2}$	<p>Subtracting fractions is easy when you have a common denominator! It's best to leave the bottom alone, since it is factored. The top, however, consists of two separate factored pieces, and will be simpler if we multiply them out so we can combine them.</p>
$= \frac{3x^2-6x-(4x^2+24x)}{x(x-2)(x+6)^2}$	<p>A common student mistake here is forgetting the parentheses. The entire second term is subtracted; without the parentheses, the $24x$ ends up being added.</p>
$= \frac{-x^2-30x}{x(x-2)(x+6)^2}$	<p>Almost done! But finally, we note that we can factor the top again. If we factor out an x it will cancel with the x in the denominator.</p>

$$= \frac{-x-30}{(x-2)(x+6)^2}$$

A lot simpler than where we started, isn't it?

The problem is long, and the math is complicated. So after following all the steps, it's worth stepping back to realize that even this problem results simply from the two rules we started with.

First, always factor rational expressions before doing anything else.

Second, follow the regular processes for fractions: in this case, the procedure for subtracting fractions, which involves finding a common denominator. After that, you subtract the numerators while leaving the denominator alone, and then simplify.

Rational Equations

This module introduces rational expressions in equations.

Rational Equations

A **rational equation** means that you are setting two rational **expressions** equal to each other. The goal is to solve for x ; that is, find the x value(s) that make the equation true.

Suppose I told you that:

Equation:

$$\frac{x}{8} = \frac{3}{8}$$

If you think about it, the x in this equation has to be a 3. That is to say, if $x=3$ then this equation is true; for any other x value, this equation is false.

This leads us to a very general rule.

A very general rule about rational equations

If you have a rational equation where the **denominators** are the same, then the **numerators** must be the same.

This in turn suggests a strategy: find a common denominator, and then set the numerators equal.

Example: Rational Equation

Example: Rational Equation

$$\frac{3}{x^2+12x+36} = \frac{4x}{x^3+4x^2-12x}$$

Same problem we worked before, but now we are equating these two fractions, instead of subtracting them.

$$\frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} = \frac{4x(x+6)}{x(x+6)^2(x-2)}$$

Rewrite both fractions with the common denominator.

$$3x(x-2) = 4x(x+6)$$

Based on the rule above—since the denominators are equal, we can now assume the numerators are equal.

$$3x^2 - 6x = 4x^2 + 24x$$

Multiply it out

$$x^2 + 30x = 0$$

What we're dealing with, in this case, is a quadratic equation. As always, move everything to one side...

$$x(x+30) = 0$$

...and then factor. A common mistake in this kind of problem is to divide both sides by x ; this loses one of the two solutions.

$$x=0 \text{ or } x=-30$$

Two solutions to the quadratic equation. However, in this case, $x = 0$ is not valid, since it was not in the domain of the original right-hand fraction. (Why?) So this problem actually has only one solution, $x = -30$.

As always, it is vital to remember what we have found here. We started with the equation $\frac{3}{x^2+12x+36} = \frac{4x}{x^3+4x^2-12x}$. We have concluded now that if you plug $x = -30$ into that equation, you will get a true equation (you can verify this on your calculator). For any other value, this equation will evaluate false.

To put it another way: if you graphed the functions $\frac{3}{x^2+12x+36}$ and $\frac{4x}{x^3+4x^2-12x}$, the two graphs would intersect at one point only: the point when $x = -30$.

Rational Expression Concepts -- Dividing Polynomials

Simplifying, multiplying, dividing, adding, and subtracting rational expressions are all based on the basic skills of working with fractions. Dividing polynomials is based on an even earlier skill, one that pretty much everyone remembers with horror: long division.

To refresh your memory, try dividing $\frac{745}{3}$ by hand. You should end up with something that looks something like this:

$$\begin{array}{r} 248 \text{ r}1 \\ 3 \overline{)745} \\ \underline{6} \\ 14 \\ \underline{12} \\ 25 \\ \underline{24} \\ 1 \end{array}$$

So we conclude that $\frac{745}{3}$ is 248 with a remainder of 1; or, to put it another way, $\frac{745}{3} = 248\frac{1}{3}$.

You may have decided years ago that you could forget this skill, since calculators will do it for you. But now it comes roaring back, because here is a problem that your calculator will not solve for you: $\frac{6x^3-8x^2+4x-2}{2x-4}$. You can solve this problem in much the same way as the previous problem.

Example:

$$\frac{6x^3-8x^2+4x-2}{2x-4}$$

The problem

$2x-4 \overline{)6x^3-8x^2+4x-2}$	<p>The problem, written in standard long division form.</p>
$\begin{array}{r} 3x^2 \\ 2x-4 \overline{)6x^3-8x^2+4x-2} \end{array}$	<p>Why $3x^2$? This comes from the question: “How many times does $2x$ go into $6x^3$?” Or, to put the same question another way: “What would I multiply $2x$ by, in order to get $6x^3$?” This is comparable to the first step in our long division problem: “What do I multiply 3 by, to get 7?”</p>
$\begin{array}{r} 3x^2 \\ 2x-4 \overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2 \end{array}$	<p>Now, multiply the $3x^2$ times the $(2x-4)$ and you get $6x^3-12x^2$. Then subtract this from the line above it. The $6x^3$ terms cancel—that shows we picked the right term above! Note that you have to be careful with signs here. $-8x^2 - (-12x^2)$ gives us positive $4x^2$.</p>
$\begin{array}{r} 3x^2 \\ 2x-4 \overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \end{array}$	<p>Bring down the $4x$. We have now gone through all four steps of long division—divide, multiply, subtract, and bring down. At this point, the process begins again, with the question “How many times does $2x$ go into $4x^2$?”</p>
$\begin{array}{r} 3x^2+2x+6 \text{ r } 22 \\ 2x-4 \overline{)6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \\ \underline{4x^2+8x} \\ 12x-2 \\ \underline{12x-24} \\ 22 \end{array}$	<p>This is not the next step...this is what the process looks like after you’ve finished all the steps. You should try going through it yourself to make sure it ends up like this.</p>

Polynomial Division

So we conclude that $\frac{6x^3-8x^2+4x-2}{2x-4}$ is $3x^2 + 2x + 6$ with a remainder of 22, or, to put it another way, $3x^2 + 2x + 6 + \frac{22}{2x-4}$.

Checking your answers

As always, checking your answers is not just a matter of catching careless errors: it is a way of making sure that you **know what you have come up with**. There are two different ways to check the answer to a division problem, and both provide valuable insight

The first is by plugging in numbers. We have created an algebraic generalization:

Equation:

$$\frac{6x^3 - 8x^2 + 4x - 2}{2x - 4} = 3x^2 + 2x + 6 + \frac{22}{2x - 4}$$

In order to be valid, this generalization must hold for $x = 3$, $x = -4$, $x = 0$, $x = \varpi$, or any other value except $x = 2$ (which is outside the domain). Let's try $x = 3$.

Checking the answer by plugging in $x = 3$

Equation:

$$\frac{6(3)^3 - 8(3)^2 + 4(3) - 2}{2(3) - 4} \stackrel{?}{=} 3(3)^2 + 2(3) + 6 + \frac{22}{2(3) - 4}$$

Equation:

$$\frac{162 - 72 + 12 - 2}{6 - 4} \stackrel{?}{=} 27 + 6 + 6 + \frac{22}{6 - 4}$$

Equation:

$$\frac{100}{2} \stackrel{?}{=} 39 + \frac{22}{2}$$

Equation:

$$50 \stackrel{?}{=} 39 + 11\checkmark$$

The second method is by multiplying back. Remember what division is: it is the opposite of multiplication! If $\frac{745}{3}$ is 248 with a remainder of 1, that means that $248 \cdot 3$ will be 745, with 1 left over. Similarly, if our long division was correct, then $(3x^2 + 2x + 6)(2x - 4) + 22$ should be $6x^3 - 8x^2 + 4x - 2$.

Checking the answer by multiplying back

Equation:

$$(3x^2 + 2x + 6)(2x - 4) + 22$$

Equation:

$$= (6x^3 - 12x^2 + 4x^2 - 8x - 24) + 22$$

Equation:

$$= 6x^3 - 8x^2 + 4x - 2\checkmark$$

Radical Concepts

This module provides an introduction to radical numbers.

The concept of a radical (or root) is a familiar one, and was reviewed in the conceptual explanation of logarithms in the previous chapter. In this chapter, we are going to explore some possibly unfamiliar properties of radicals, and solve equations involving radicals.

Radical Concepts -- Properties of Radicals

What is $\sqrt{x^2 + 9}$? Many students will answer quickly that the answer is $(x + 3)$ and have a very difficult time believing this answer is wrong. But it is wrong.

$\sqrt{x^2}$ is x^* [\[footnote\]](#) and $\sqrt{9}$ is 3, but $\sqrt{x^2 + 9}$ is **not** $(x + 3)$.

I'm fudging a bit here: $\sqrt{x^2}$ is x only if you ignore negative numbers. For instance, if $x = -3$, then $x^2 = 9$, and $\sqrt{x^2}$ is 3; so in that case, $\sqrt{x^2}$ is not x . In general, $\sqrt{x^2} = |x|$. However, this subtlety is not relevant to the overall point, which is that you cannot break up two terms that are added under a radical.

Why not? Remember that $\sqrt{x^2 + 9}$ is asking a question: "what **squared** gives the answer $x^2 + 9$?" So $(x + 3)$ is not an answer, because $(x + 3)^2 = x^2 + 6x + 9$, **not** $x^2 + 9$.

As an example, suppose $x = 4$. So $\sqrt{x^2 + 9} = \sqrt{4^2 + 9} = \sqrt{25} = 5$. But $(x + 3) = 7$.

Note: If two numbers are **added** or **subtracted** under a square root, you cannot split them up. In symbols: $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$ or, to put it another way, $\sqrt{x^2 + y^2} \neq x + y$

$\sqrt{x^2 + 9}$ cannot, in fact, be simplified at all. It is a perfectly valid function, but cannot be rewritten in a simpler form.

How about $\sqrt{9x^2}$? By analogy to the previous discussion, you might expect that this cannot be simplified either. But in fact, it can be simplified:

$$\sqrt{9x^2} = 3x$$

Why? Again, $\sqrt{9x^2}$ is asking “what **squared** gives the answer $9x^2$?” The answer is $3x$ because $(3x)^2 = 9x^2$.

Similarly, $\sqrt{\frac{9}{x^2}} = \frac{3}{x}$, because $(\frac{3}{x})^2 = \frac{9}{x^2}$.

Note: If two numbers are multiplied or divided under a square root, you **can** split them up. In symbols: $\sqrt{ab} = \sqrt{a}\sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Simplifying Radicals

This module covers techniques for the simplification of radicals.

Simplifying Radicals

The property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ can be used to simplify radicals. The key is to break the number inside the root into two factors, **one of which is a perfect square**.

Example: Simplifying a Radical

$\sqrt{75}$	
$= \sqrt{25 \bullet 3}$	because 25•3 is 75, and 25 is a perfect square
$= \sqrt{25}\sqrt{3}$	because $\sqrt{ab} = \sqrt{a}\sqrt{b}$
$= 5\sqrt{3}$	because $\sqrt{25} = 5$

So we conclude that $\sqrt{75} = 5\sqrt{3}$. You can confirm this on your calculator (both are approximately 8.66).

We rewrote 75 as $25 \bullet 3$ because 25 is a perfect square. We could, of course, also rewrite 75 as $5 \bullet 15$, but—although correct—that would not help us simplify, because neither number is a perfect square.

Example:
Simplifying a Radical in Two Steps

$\sqrt{180}$	
$= \sqrt{9 \bullet 20}$	because $9 \bullet 20$ is 180, and 9 is a perfect square
$= \sqrt{9} \sqrt{20}$	because $\sqrt{ab} = \sqrt{a} \sqrt{b}$
$= 3 \sqrt{20}$	So far, so good. But wait! We're not done!
$= 3 \sqrt{4 \bullet 5}$	There's another perfect square to pull out!
$= 3 \sqrt{4} \sqrt{5}$	
$= 3(2) \sqrt{5}$	
$= 6 \sqrt{5}$	Now we're done.

The moral of this second example is that **after** you simplify, you should always look to see if you can simplify **again** .

A secondary moral is, try to pull out the biggest perfect square you can. We could have jumped straight to the answer if we had begun by rewriting 180 as $36 \bullet 5$.

This sort of simplification can sometimes allow you to **combine** radical terms, as in this example:

**Example:
Combining Radicals**

$\sqrt{75} - \sqrt{12}$	
$= 5\sqrt{3} - 2\sqrt{3}$	We found earlier that $\sqrt{75} = 5\sqrt{3}$. Use the same method to confirm that $\sqrt{12} = 2\sqrt{3}$.
$= 3\sqrt{3}$	5 of anything minus 2 of that same thing is 3 of it, right?

That last step may take a bit of thought. It can only be used when the radical is the same. Hence, $\sqrt{2} + \sqrt{3}$ cannot be simplified at all. We were able to simplify $\sqrt{75} - \sqrt{12}$ only by making the radical in both cases the same .

So why does $5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$? It may be simplest to think about verbally: 5 of these things, minus 2 of the same things, is 3 of them. But you can look at it more formally as a factoring problem, if you see a common factor of $\sqrt{3}$.

$$5\sqrt{3} - 2\sqrt{3} = \sqrt{3}(5 - 2) = \sqrt{3}(3).$$

Of course, the process is exactly the same if variable are involved instead of just numbers!

Example:
Combining Radicals with Variables

$x^{\frac{3}{2}} + x^{\frac{5}{2}}$	
$= x^3 + x^5$	Remember the definition of fractional exponents!
$= \sqrt{x^2 * x} + \sqrt{x^4 * x}$	As always, we simplify radicals by factoring them inside the root...
$\sqrt{x^2 * x} + \sqrt{x^4 * x}$	and then breaking them up...
$= x\sqrt{x} + x^2\sqrt{x}$	and then taking square roots outside!
$= (x^2 + x)\sqrt{x}$	Now that the radical is the same, we can combine.

Rationalizing the Denominator

It is always possible to express a fraction with no square roots in the denominator.

Is it always desirable? Some texts are religious about this point: "You should never have a square root in the denominator." I have absolutely no idea why. To me, $\frac{1}{\sqrt{2}}$ looks simpler than $\frac{\sqrt{2}}{2}$; I see no overwhelming reason for forbidding the first or preferring the second.

However, there are times when it is useful to remove the radicals from the denominator: for instance, when adding fractions. The trick for doing this is based on the basic rule of fractions: **if you multiply the top and bottom of a fraction by the same number, the fraction is unchanged.** This rule enables us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$.

In a case like $\frac{1}{\sqrt{2}}$, therefore, you can multiply the top and bottom by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$$

What about a more complicated case, such as $\frac{\sqrt{12}}{1+\sqrt{3}}$? You might think we could simplify this by multiplying the top and bottom by $(1+\sqrt{3})$, but that doesn't work: the bottom turns into $(1+\sqrt{3})^2 = 1 + 2\sqrt{3} + 3$, which is at least as ugly as what we had before.

The correct trick for getting rid of $(1+\sqrt{3})$ is to multiply it by $(1-\sqrt{3})$. These two expressions, identical except for the replacement of a+ by a-, are known as **conjugates**. What happens when we multiply them? We don't need to use FOIL if we remember that

$$(x+y)(x-y) = x^2 - y^2$$

Using this formula, we see that

$$(1+\sqrt{3})(1-\sqrt{3}) = 1^2 - (\sqrt{3})^2 = 1 - 3 = -2$$

So the square root does indeed go away. We can use this to simplify the original expression as follows.

Example:

Rationalizing Using the Conjugate of the Denominator

$$\frac{\sqrt{12}}{1+\sqrt{3}} = \frac{\sqrt{12}(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{\sqrt{12}-\sqrt{36}}{1-3} = \frac{2\sqrt{3}-6}{-2} = -\sqrt{3} + 3$$

As always, you may want to check this on your calculator. Both the original and the simplified expression are approximately 1.268.

Of course, the process is the same when variables are involved.

Example:

Rationalizing with Variables

$$\frac{1}{x-\sqrt{x}} = \frac{1(x+\sqrt{x})}{(x-\sqrt{x})(x+\sqrt{x})} = \frac{x+\sqrt{x}}{x^2-x}$$

Once again, we multiplied the top and the bottom by the **conjugate of the denominator**: that is, we replaced a- with a+. The formula $(x+a)(x-a) = x^2 - a^2$ enabled us to quickly multiply the terms on the bottom, and eliminated the square roots in the denominator.

Radical Equations

This module contains methods on solving radical equations.

When solving **equations that involve radicals**, begin by asking yourself: **is there an x under the square root?** The answer to this question will determine the way you approach the problem.

If there is **not** an x under the square root—if only numbers are under the radicals—you can solve much the same way you would solve with no radicals at all.

Example:

Radical Equation with No Variables Under Square Roots

$\sqrt{2}x + 5 = 7 - \sqrt{3}x$	Sample problem: no variables under radicals
$\sqrt{2} + \sqrt{3}x = 7 - 5$	Get everything with an x on one side, everything else on the other
$x(\sqrt{2} + \sqrt{3}) = 2$	Factor out the x
$x = \frac{2}{\sqrt{2} + \sqrt{3}}$	Divide, to solve for x

The key thing to note about such problems is that **you do not have to square both sides of the equation**. $\sqrt{2}$ may look ugly, but it is just a number—you could find it on your calculator if you wanted to—it functions in the equation just the way that the number 10, or $\frac{1}{3}$, or π would.

If there is an x under the square root, the problem is completely different. You will have to square both sides to get rid of the radical. However, there are two important notes about this kind of problem.

1. Always get the radical **alone, on one side of the equation**, before squaring.
2. Squaring both sides can introduce **false answers**—so it is important to check your answers after solving!

Both of these principles are demonstrated in the following example.

Example:

Radical Equation with Variables under Square Roots

$\sqrt{x + 2} + 3x = 5x + 1$	Sample problem with variables under radicals
$\sqrt{x + 2} = 2x + 1$	Isolate the radical before squaring!
$x + 2 = (2x + 1)^2$	Now, square both sides
$x + 2 = 4x^2 + 4x + 1$	Multiply out. Hey, it looks like a quadratic equation now!
$x + 2 = 4x^2 + 4x + 1$	As always with quadratics, get everything on one side.
$(4x - 1)(x + 1) = 0$	Factoring: the easiest way to solve quadratic equations.
$x = \frac{1}{4}$ or $x = -1$	Two solutions. Do they work? Check in the original equation!
Check $x = \frac{1}{4}$	Check $x = -1$

$\sqrt{\frac{1}{4} + 2} + 3\left(\frac{1}{4}\right) \stackrel{?}{=} 5\left(\frac{1}{4}\right) + 1$	$\sqrt{-1 + 2} + 3(-1) \stackrel{?}{=} 5(-1) + 1$
$\sqrt{\frac{1}{4} + \frac{8}{4}} + \frac{3}{4} \stackrel{?}{=} \frac{5}{4} + 1$	$\sqrt{1} - 3 \stackrel{?}{=} -5 + 1$
$\sqrt{\frac{9}{4}} + \frac{3}{4} \stackrel{?}{=} \frac{5}{4} + \frac{4}{4}$	$1 - 3 \stackrel{?}{=} -5 + 1$
$\frac{3}{2} + \frac{3}{4} \stackrel{?}{=} \frac{5}{4} + \frac{4}{4}$	$-2 = -4$ Not equal!
$\frac{9}{4} = \frac{9}{4}$	

So the algebra yielded two solutions: $\frac{1}{4}$ and -1 . Checking, however, we discover that only the first solution is valid. This problem demonstrates how important it is to check solutions whenever squaring both sides of an equation.

If variables under the radical occur more than once, you will have to go through this procedure multiple times. Each time, you isolate a radical and then square both sides.

Example:	
Radical Equation with Variables under Square Roots Multiple Times	
$\sqrt{x + 7} - x = 1$	Sample problem with variables under radicals multiple times
$\sqrt{x + 7} = \sqrt{x} + 1$	Isolate one radical. (I usually prefer to start with the bigger one.)

$x + 7 = x + 2\sqrt{x} + 1$	Square both sides. The two -radical equation is now a one -radical equation.
$6 = 2\sqrt{x}$	
$3 = x$	Isolate the remaining radical, then square both sides again..
$9 = x$	In this case, we end up with only one solution. But we still need to check it.
$\sqrt{9+7} - \sqrt{9} \stackrel{?}{=} 1$	
$\sqrt{16} - \sqrt{9} \stackrel{?}{=} 1$	
$4 - 3 = 1$	
Check x=9	

Remember, the key to this problem was recognizing that **variables under the radical** occurred in the original problem **two times**. That cued us that we would have to go through the process— isolate a radical, then square both sides—twice, before we could solve for x . And **whenever** you square both sides of the equation, it's vital to check your answer(s)!

When good math leads to bad answers

Why is it that—when squaring both sides of an equation—perfectly good algebra can lead to invalid solutions? The answer is in the redundancy of squaring. Consider the following equation:

$-5 = 5$ False. But square both sides, and we get...

$25 = 25$ True. So squaring both sides of a **false** equation can produce a **true** equation.

To see how this affects our equations, try plugging $x = -1$ into the various steps of the first example.

Example:

Why did we get a false answer of $x=-1$ in Example 1?

$\sqrt{x+2} + 3x = 5x + 1$	Does $x = -1$ work here? No, it does not.
$\sqrt{x+2} = 2x + 1$	How about here? No, $x = -1$ produces the false equation $1=-1$.
$x + 2 = (2x + 1)^2$	Suddenly, $x = -1$ works. (Try it!)

When we squared both sides, we “lost” the difference between 1 and -1 , and they “became equal.” From here on, when we solved, we ended up with $x = -1$ as a valid solution.

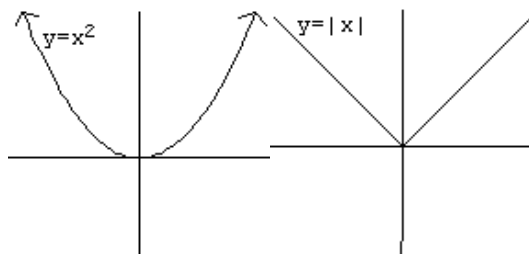
Test your memory: When you square both sides of an equation, you can introduce false answers. We have encountered one other situation where **good algebra** can lead to a **bad answer**. When was it?

Answer: It was during the study of absolute value equations, such as $|2x + 3| = -11x + 42$. In those equations, we also found the hard-and-fast rule that you **must check your answers** as the last step.

What do these two types of problem have in common? The function $|x|$ actually has a lot in common with x^2 . Both of them have the peculiar property that they

always turn $-a$ and a into the same response. (For instance, if you plug -3 and 3 into the function, you get the same thing back.) This property is known as being an **even function**. Dealing with such “redundant” functions leads, in both cases, to the possibility of false answers.

The similarity between these two functions can also be seen in the graphs: although certainly not identical, they bear a striking resemblance to each other. In particular, both graphs are symmetric about the y -axis, which is the fingerprint of an “even function”.



Imaginary Numbers Concepts

This module introduces i , the imaginary number, and the concept of complex numbers in Algebra.

$$(-1)^2 = 1$$

$$1^2 = 1$$

Whether you square a positive or a negative number, the answer is positive. It is impossible to square **any** number and get a negative answer.

So what is $\sqrt{-1}$? Since it asks the question “What number **squared** is -1 ?”, and since nothing squared ever gives the answer -1 , we say that the question has no answer. More generally, we say that the domain of \sqrt{x} is all numbers x such that $x \geq 0$. -1 is not in the domain.

However, it turns out that for a certain class of problems, it is useful to define a new kind of number that has the peculiar property that when you square them, you **do** get negative answers.

Definition of i

The definition of the imaginary number i is that it is the square root of -1 :

$$i = \sqrt{-1} \text{ or, equivalently, } i^2 = -1$$

i is referred to as an “imaginary number” because it cannot represent real quantities such as “the number of rocks” or “the length of a stick.”

However, surprisingly, imaginary numbers can be useful in solving many real world problems!

I often like to think of x as being like a science fiction story. Many science fiction stories are created by starting with **one false premise**, such as “time travel is possible” or “there are men on Mars,” and then following that premise logically to see where it would lead. With imaginary numbers, we start with the premise that “a number exists whose square is -1 .” We then follow that premise logically, using all the established rules of math, to see where it leads.

" "The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being." " - Gottfried Wilhelm Leibniz"

Playing with i

This module contains some example problems involving the manipulation i , the imaginary number.

Let's begin with a few very simple exercises designed to show how we apply the normal rules of algebra to this new, abnormal number.

A few very simple examples of expressions involving i	
Simplify:	$i \bullet 5$
Answer:	$5i$
Simplify:	$i + 5i$
Answer:	$6i$ (Add anything to 5 of itself, and you get 6 of it. Or, you can think of this as "pulling out" an i as follows: $i + 5i = i(1 + 5) = 6i$)
Simplify:	$2i + 3$
Answer:	You can't simplify it.

Now let's try something a little more involved.

Example: Simplify the expression $(3+2i)^2$	
$(3 + 2i)^2 = 3^2 + 2(3)(2i) + (2i)^2$	because $(x + a)^2 = x^2 + 2ax + a^2$ as always
$= 9 + 12i - 4$	$(2i)^2 = (2i)(2i) = (2)(2)(i)(i) = 4i^2 = -4$
$= 5 + 12i$	we can combine the 9 and -4, but not the 12i.

It is vital to remember that i is not a variable, and this is not an algebraic generalization. You cannot plug $i = 3$ into that equation and expect anything valid to come out. The equation $(3+2i)^2 = 5 + 12i$ has been shown to be true for only one number: that number is i , the square root of -1 .

In the next example, we simplify a radical using **exactly the same technique that we used in the unit on radicals**, except that $a-1$ is thrown into the picture.

Example: Simplify $\sqrt{-20}$	
$\sqrt{-20} = \sqrt{(4)(5)(-1)}$	as always, factor out the perfect squares
$= \sqrt{4}\sqrt{5}\sqrt{-1}$	then split it, because $\sqrt{ab} = \sqrt{a}\sqrt{b}$
$= 2i\sqrt{5}$	$\sqrt{4}=2$, $\sqrt{-1}= i$, and $\sqrt{5}$ is just $\sqrt{5}$
Check	
Is $2i\sqrt{5}$ really the square root of -20 ? If it is, then when we square it, we should get -20 .	
$(2i\sqrt{5})^2 = 2^2 i^2 5^2 = 4 * -1 * 5 = -20$	
<input checked="" type="checkbox"/> It works!	

The problem above has a very important consequence. We began by saying “You can’t take the square root of any negative number.” Then we defined i as the square root of -1 . But we see that, using i , we can now take the square root of **any** negative number.

Complex Numbers

This module introduces the concept of complex numbers in Algebra.

A “complex number” is the sum of two parts: a real number by itself, and a real number multiplied by i . It can therefore be written as $a + bi$, where a and b are real numbers.

The first part, a , is referred to as the **real part**. The second part, bi , is referred to as the **imaginary part**.

Examples of complex numbers $a + bi$ (a is the “real part”; bi is the “imaginary part”)	
$3 + 2i$	$a = 3, b = 2$
π	$a = \pi, b = 0$ (no imaginary part: a “pure real number”)
$-i$	$a = 0, b = -1$ (no real part: a “pure imaginary number”)

Some numbers are not obviously in the form $a + bi$. However, any number can be **put** in this form.

Example 1: Putting a fraction into $a + bi$ form (i in the numerator)
$\frac{3-4i}{5}$ is a valid complex number. But it is not in the form $a + bi$, and we cannot immediately see what the real and imaginary parts are.

Example 1: Putting a fraction into $a + bi$ form (i in the numerator)

To see the parts, we rewrite it like this:

$$\frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i$$

Why does that work? It's just the ordinary rules of fractions, applied backward. (Try multiplying and then subtracting on the right to confirm this.) But now we have a form we can use:

$$\frac{3-4i}{5} \quad a = \frac{3}{5}, \quad b = -\frac{4}{5}$$

So we see that fractions are very easy to break up, if the i is in the numerator. An i in the denominator is a bit trickier to deal with.

Example 2: Putting a fraction into $a + bi$ form (i in the denominator)

$$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i}$$

Multiplying the top and bottom of a fraction by the same number never changes the value of the fraction: it just rewrites it in a different form.

$$= \frac{i}{-1}$$

Because $i \cdot i$ is i^2 , or -1 .

$$= -i$$

This is not a property of i , but of -1 . Similarly, $\frac{5}{-1} = -5$.

Example 2: Putting a fraction into $a + bi$ form (i in the denominator)

$$\frac{1}{i}:$$

$$a = 0,$$

$$b = -1$$

since we rewrote it as $-i$, or $0 - 1i$

Finally, what if the denominator is a more complicated complex number? The trick in this case is similar to the trick we used for rationalizing the denominator: we multiply by a quantity known as the **complex conjugate of the denominator**.

Definition of Complex Conjugate

The complex conjugate of the number $a + bi$ is $a - bi$. In words, you leave the real part alone, and change the sign of the imaginary part.

Here is how we can use the “complex conjugate” to simplify a fraction.

Example: Using the Complex Conjugate to put a fraction into $a + bi$ form

$$\frac{5}{3-4i}$$

The fraction: a complex number not currently in the form $a + bi$

$$= \frac{5(3+4i)}{(3-4i)(3+4i)}$$

Multiply the top and bottom by the complex conjugate of the denominator

$$= \frac{15+20i}{3^2-(4i)^2}$$

Remember, $(x + a)(x - a) = x^2 - a^2$

Example: Using the Complex Conjugate to put a fraction into $a + bi$ form

$$= \frac{15+20i}{9+16}$$

$(4i)^2 = 4^2i^2 = 16(-1) = -16$, which we are subtracting from 9

$$= \frac{15+20i}{25}$$

Success! The top has i , but the bottom doesn't. This is easy to deal with.

$$= \frac{15}{25} + \frac{20i}{25}$$

Break the fraction up, just as we did in a previous example.

$$= \frac{3}{5} + \frac{4}{5}i$$

So we're there! $a = \frac{3}{5}$ and $b = \frac{4}{5}$

Any number of any kind can be written as $a + bi$. The above examples show how to rewrite fractions in this form. In the text, you go through a worksheet designed to rewrite $\sqrt[3]{-1}$ as three different complex numbers. Once you understand this exercise, you can rewrite other radicals, such as \sqrt{i} , in $a + bi$ form.

Imaginary Concepts -- Equality and Inequality in Complex Numbers

What does it mean for two complex numbers to be **equal**? As always, equality asserts that two things are exactly the same. $7 + 3i$ is **not** equal to 7, or to $3i$, or to $7 - 3i$, or to $3 + 7i$. It is not equal to anything except $7 + 3i$.

Definition of Equality

Two complex numbers are **equal** to each other only if **their real parts are equal, and their imaginary parts are equal**.

So if we say that two complex numbers equal each other, we are actually making two separate, independent statements. We can use this, for instance, to solve for two separate variables.

Example: Complex Equality	
If $3x + 4yi + 7 = 4x + 8i$, what are x and y ?	
Normally, it is impossible to solve one equation for two unknowns. But this is really two separate equations!	
Real part on the left = real part on the right:	$3x + 7 = 4x$
Imaginary part on the left = imaginary part on the right:	$4y = 8$
We can now solve both of these equations trivially.	$x = 7,$ $y = 2$

And what about **inequalities**? The answer may surprise you: **there are no inequalities with complex numbers**, at least not in the form we're seeing.

The real numbers have the property that for any two real numbers a and b , exactly one of the following three statements must be true: $a = b$, $a > b$, or $a < b$. This is one of those properties that seems almost too obvious to bother with. But it becomes more interesting when you realize that the complex numbers do **not** have that property. Consider two simple numbers, i and 1 . Which of the following is true?

- $i = 1$



- $i > 1$



- $i < 1$



None of them is true. It is not generally possible to describe two complex numbers as being “greater than” or “less than” each other.

Visually, this corresponds to the fact that all the real numbers can be laid out on a number line: “greater than” means “to the right of” and so on. The complex numbers cannot be laid out on a number line. They are sometimes pictured on a 2-dimensional graph, where the real part is the x coordinate and the imaginary part is the y coordinate. But one point on a graph is neither greater than, nor less than, another point!

Quadratic Equations and Complex Numbers

This module looks at quadratic equations with a negative discriminant in Algebra.

In the unit on quadratic equations and complex numbers, we saw that a quadratic equation can have **two answers, one answer, or no answers**.

We can now modify this third case. In cases where we described “no answers” there are actually two answers, but both are complex! This is easy to see if you remember that we found “no answers” when the **discriminant was negative**—that is, when the quadratic formula gave us a negative answer in the square root.

As an example, consider the equation:

$$2x^2 + 3x + 5 = 0$$

The quadratic equation gives us:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(5)}}{4} = \frac{-3 \pm \sqrt{-31}}{4}$$

This is the point where, in the “old days,” we would have given up and declared “no answer.” Now we can find two answers—both complex.

$$= \frac{-3}{4} \pm \frac{\sqrt{31}\sqrt{-1}}{4} = \frac{-3}{4} \pm \frac{\sqrt{31}}{4}i$$

So we have two answers. Note that the two answers are **complex conjugates** of each other—this relationship comes directly from the quadratic formula.

A Few “Extra for Experts” Thoughts on Imaginary Numbers

This is a module discussing several advanced concepts relating to complex numbers, including the differences between real and imaginary, and rational and irrational numbers.

Illegal Operations

So far, we have seen three different **illegal operations** in math.

1. You cannot take the square root of a negative number. (Hence, the domain of \sqrt{x} is $x \geq 0$.)
2. You cannot divide by zero. (Hence, the domain of $\frac{1}{x}$ is $x \neq 0$.)
3. You cannot take the log of 0 or a negative number. (Hence, the domain of $\log(x)$ is $x > 0$.)

Imaginary numbers give us a way of violating the first restriction. Less obviously, they also give us a way of violating the third restriction: with imaginary numbers, you can take the log of a negative number.

So, how about that second restriction? Do you ever reach a point in math where the teacher admits “OK, we **really can** divide by 0 now”? Can we define a new imaginary number $j = \frac{1}{0}$?

The answer is emphatically no: **you really can’t divide by 0**. If you attempt to define an imaginary way around this problem, all of math breaks down. Consider the following simple example:

$5 \bullet 0 = 3 \bullet 0$	That’s true
$5 = 3$	Divide both sides by 0

You see? Dividing by 0 takes us from true conclusions to false ones.

The astonishing thing about the definition $i = \sqrt{-1}$ is that, although it is imaginary and nonsensical, it is **consistent**: it does not lead to any logical contradictions. You can find many ways to simplify $\frac{1}{i}$ and it will always reduce to $-i$ in the end. Division by zero can never be consistent in this way, so it is always forbidden.

A great deal of Calculus is concerned with getting around this problem, by dividing by numbers that are **very close** to zero.

The World of Numbers

When you first learn about numbers, you learn the **counting numbers**:

1,2,3,4...	counting numbers
------------	-------------------------

These numbers are perfect for answering questions such as “How many sticks do I have?” “How many days until Christmas?” “How many years old are you?”

For other questions, however, you run into limitations. In measuring temperature, for instance, we find that we need lower numbers than 1. Hence, we arrive at a broader list:

...-4,-3,-2,-1,0,1,2,3,4...	integers
-----------------------------	-----------------

The addition of 0 and the negative numbers gives us a new, broader set. The original idea of numbers is now seen as a special case of this more general idea; the original set is a subset of this one.

Still, if we are measuring lengths of sticks, we will find that often they fall **between** our numbers. Now we have to add fractions, or decimals, to create the set of **rational numbers**. I can no longer list the set, but I can give examples.

$\frac{1}{2}, -3, \frac{22}{7}, 0, 2.718, 0.14141414\dots$	rational numbers
--	-------------------------

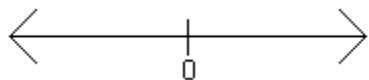
The word “rational” implies a **ratio**, or fraction: the ratio of **two integers**. Hence, we define our new, broader set (rational numbers) in terms of our older, more limited set (integers). Rational numbers can be expressed as either fractions, or as decimals (which either end after a certain number of digits, or repeat the same loop of digits forever).

This set seems to be all-inclusive, but it isn't: certain numbers cannot be expressed in this form.

$\sqrt{2}, \pi$	irrational numbers
-----------------	---------------------------

The square root of any non-perfect square is “irrational” and so is π . They can be **approximated** as fractions, but not expressed exactly. As decimals, they go on forever but do not endlessly repeat the same loop.

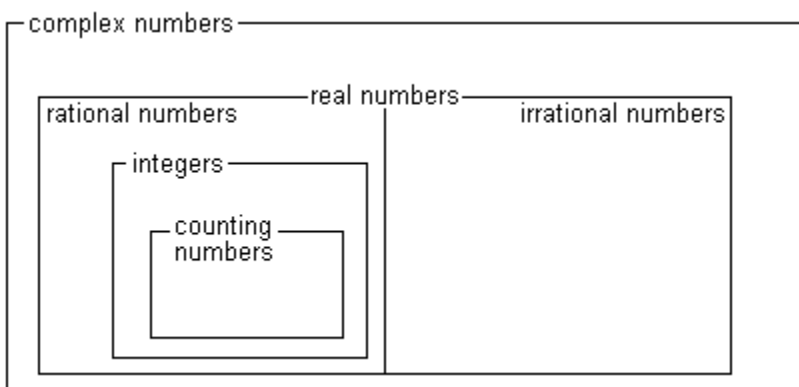
If you take the rationals and irrationals together, you get the **real numbers**. The real numbers are all the numbers represented on a number line.



All the numbers
on a number line
are the real
numbers

Now, with this unit, we have added the final piece of the puzzle, the **complex numbers**. A complex number is any number $a + bi$ where a and b are real numbers. Hence, just as our definition of rational numbers was **based on** our definition of integers, so our definition of complex numbers is **based on** our definition of real numbers. And of course, if $b = 0$ then we have a real number: the old set is a subset of the new.

All of this can be represented in the following diagram.



The diagram captures the vital idea of subsets: all real numbers are complex numbers, but not all complex numbers are real.

Similarly, the diagram shows that if you take all the rational numbers, and all the irrational numbers, together they make up the set of real numbers.

Matrices

This module introduces basic properties of matrices: concepts, multiplication by a constant, addition and subtraction, and setting two matrices equal to one another.

Conceptual Explanations: Matrices

A “matrix” is a grid, or table, of numbers. For instance, the following matrix represents the prices at the store “Nuthin’ But Bed Stuff.”

	King-sized	Queen-sized	Twin
Mattress	\$649	\$579	\$500
Box spring	\$350	\$250	\$200
Fitted sheet	\$15	\$12	\$10
Top sheet	\$15	\$12	\$10
Blanket	\$20	\$20	\$15

(The matrix is the numbers, not the words that label them.)

Of course, these prices could be displayed in a simple list: “King-sized mattress,” “Queen-sized mattress,” and so on. However, this two-dimensional display makes it much easier to compare the prices of mattresses to box springs, or the prices of king-sized items to queen-sized items, for instance.

Each horizontal list of numbers is referred to as a **row**; each vertical list is a **column**. Hence, the list of all mattresses is a row; the list of all king-sized

prices is a column. (It's easy to remember which is which if you think of Greek columns, which are big posts that hold up buildings and are very tall and...well, you know...vertical.) This particular matrix has 5 rows and 3 columns. It is therefore referred to as a 5×3 (read, "5 by 3") matrix.

If a matrix has the same number of columns as rows, it is referred to as a **square matrix**.

Adding and Subtracting Matrices

Adding matrices is very simple. You just add each number in the first matrix, to the corresponding number in the second matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 61 & 52 & 43 \\ 34 & 25 & 16 \end{bmatrix}$$

For instance, for the upper-right-hand corner, the calculation was $3 + 40 = 43$. Note that both matrices being added are 2×3 , and the resulting matrix is also 2×3 . **You cannot add two matrices that have different dimensions.**

As you might guess, subtracting works much the same way, except that you subtract instead of adding.

$$\begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 59 & 48 & 37 \\ 26 & 15 & 4 \end{bmatrix}$$

Once again, note that the resulting matrix has the same dimensions as the originals, and that **you cannot subtract two matrices that have different dimensions.**

Multiplying a Matrix by a Constant

What does it mean to multiply a number by 3? It means you add the number to itself 3 times.

Multiplying a **matrix** by 3 means the same thing...you add the matrix to itself 3 times.

$$3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

Note what has happened: each element in the original matrix has been multiplied by 3. Hence, we arrive at the method for multiplying a matrix by a constant: you multiply each element by that constant. The resulting matrix has the same dimensions as the original.

$$\frac{1}{2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \end{bmatrix}$$

Matrix Equality

For two matrices to be “equal” they must be exactly the same. That is, they must have the same dimensions, and each element in the first matrix must be equal to the corresponding element in the second matrix.

For instance, consider the following matrix equation.

$$\begin{bmatrix} 1 & + \\ 12 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ - & 10 \end{bmatrix}$$

Both matrices have the same dimensions. And the upper-left and lower-right elements are definitely the same.

But for the matrix to be equal, we also need the other two elements to be the same. So

$$+ = 18$$

$$- = 12$$

Solving these two equations (for instance, by elimination) we find that
 $= 15, = 3.$

You may notice an analogy here to complex numbers. When we assert that two complex numbers equal each other, we are actually making two statements: the **real** parts are equal, and the **imaginary** parts are equal. In such a case, we can use one equation to solve for two unknowns. A very similar situation exists with matrices, except that one equation actually represents **many more** statements. For 2×2 matrices, setting them equal makes four separate statements; for 2×3 matrices, six separate statements; and so on.

OK, take a deep breath. Even if you've never seen a matrix before, the concept is not too difficult, and everything we've seen so far should be pretty simple, if not downright obvious.

Let that breath out now. This is where it starts to get weird.

Multiplying Matrices

This module covers multiplication of matrices.

Multiplying a Row Matrix by a Column Matrix

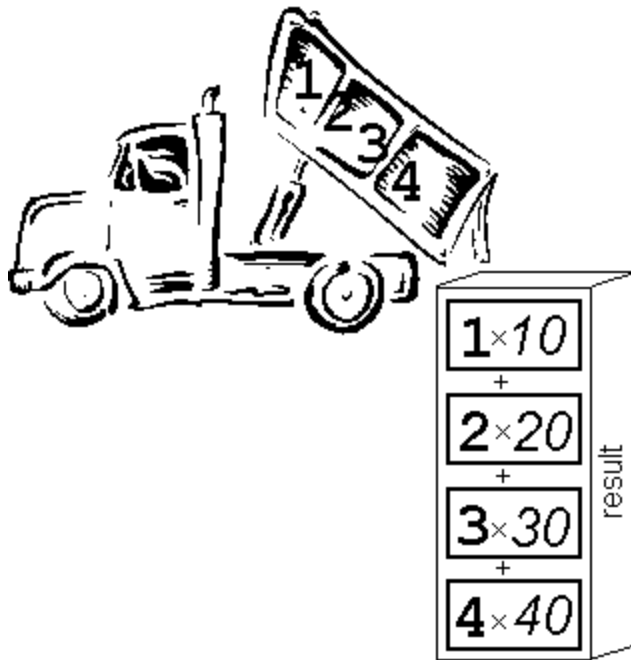
A “row matrix” means a matrix with only one row. A “column matrix” means a matrix with only one column. When a row matrix has the same number of elements as a column matrix, they can be multiplied. So the following is a perfectly **legal** matrix multiplication problem:

$$\begin{array}{cccc} & & & 10 \\ & & & 20 \\ [1 & 2 & 3 & 4] \times & & 30 \\ & & & 40 \end{array}$$

These two matrices could not be added, of course, since their dimensions are different, but they **can** be multiplied. Here’s how you do it. You multiply the first (left-most) item in the row, by the first (top) item in the column. Then you do the same for the second items, and the third items, and so on. Finally, you **add** all these products to produce the final number.

$$\begin{array}{cccc} & & & 10 \\ & & & 20 \\ & & & 30 \\ & & & 40 \end{array} \Rightarrow \begin{array}{c} 1 \cdot 10 \\ + \\ 2 \cdot 20 \\ + \\ 3 \cdot 30 \\ + \\ 4 \cdot 40 \end{array} = [300]$$

A couple of my students (Nakisa Asefnia and Laura Parks) came up with an ingenious trick for visualizing this process. Think of the row as a dump truck, backing up to the column dumpster. When the row dumps its load, the numbers line up with the corresponding numbers in the column, like so:



So, without the trucks and dumpsters, we express the result—a row matrix, times a column matrix—like this:

$$\begin{array}{cccc}
 & & & 10 \\
 & & & 20 \\
 1 & 2 & 3 & 4 \\
 & & & 30 \\
 & & & 40
 \end{array} = 300$$

There are several subtleties to note about this operation.

- The picture is a bit deceptive, because it might appear that you are multiplying two columns. In fact, **you cannot multiply a column matrix by a column matrix**. We are multiplying a row matrix by a column matrix. The picture of the **row** matrix “dumping down” only demonstrates which numbers to multiply.
- The answer to this problem is not a number: it is a 1-by-1 matrix.
- The multiplication can only be performed if the **number of elements** in each matrix is the same. (In this example, each matrix has 4 elements.)
- Order matters! We are multiplying a **row matrix times a column matrix**, not the other way around.

It's important to practice a few of these, and get the hang of it, before you move on.

Multiplying Matrices in General

The general algorithm for multiplying matrices is built on the row-times-column operation discussed above. Consider the following example:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \quad \begin{array}{cc} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{array}$$

The key to such a problem is to think of the first matrix as a list of **rows** (in this case, 4 rows), and the second matrix as a list of **columns** (in this case, 2 columns). You are going to multiply each **row** in the first matrix, by each **column** in the second matrix. In each case, you will use the “dump truck” method illustrated above.

Start at the beginning: first row, times first column.

Multiply the first row of the first matrix, times the first column of the second matrix. The result becomes the upper-left-hand corner of the answer matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 140 \\ \\ \\ \end{bmatrix}$$

$1 \cdot 10 + 2 \cdot 20 + 3 \cdot 30 = 140$

Now, move **down** to the next row. As you do so, move **down** in the answer matrix as well.

Now continue working in the same column (first column, second matrix). But begin moving down the rows of the first matrix, and moving down the rows of the answer matrix you are building.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 \\ 320 \\ 500 \\ 680 \end{bmatrix}$$

$4 \cdot 10 + 5 \cdot 20 + 6 \cdot 30 = 320$

Now, move down the rows in the first matrix, multiplying each one by that same column on the right. List the numbers below each other.

And so on, down the rows of the first matrix. Each one multiplies by the first column of the second matrix, to produce the whole first column of the answer.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 \\ 320 \\ 500 \\ 680 \end{bmatrix}$$

The **first column** of the second matrix has become the **first column** of the answer. We now move on to the **second column** and repeat the entire process, starting with the first row.

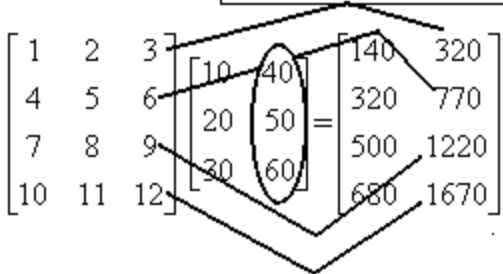
Same steps, new column. Begin with the top row of the first matrix, and it becomes the top row in the answer.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ 320 & 500 \\ 500 & 680 \\ 680 & 860 \end{bmatrix}$$

$1 \cdot 40 + 2 \cdot 50 + 3 \cdot 60 = 320$

And so on, working our way once again through all the rows in the first matrix.

And so the second column in the second matrix creates the second column in the answer. And so on...if the second matrix had 17 columns, then our answer would have 17 columns. In this case, since it has only two columns, we're done.



We're done. We can summarize the results of this entire operation as follows:

$$\begin{array}{ccc} 1 & 2 & 3 & & & 140 & 320 \\ & & & 10 & 40 & & & \\ 4 & 5 & 6 & & & 320 & 770 \\ & & & 20 & 50 & = & & \\ 7 & 8 & 9 & & & 500 & 1220 \\ & & & 30 & 60 & & & \\ 10 & 11 & 12 & & & 680 & 1670 \end{array}$$

It's a strange and ugly process—but everything we're going to do in the rest of this unit builds on this, so it's vital to be comfortable with this process.

The only way to become comfortable with this process is to do it. A lot. Multiply a lot of matrices until you are confident in the steps.

Note that we could add more rows to the first matrix, and that would add more rows to the answer. We could add more **columns** to the second matrix, and that would add more columns to the answer. However—if we added a column to the first matrix, or added a row to the second matrix, we would have an illegal multiplication. As an example, consider what happens if we try to do this multiplication in reverse:

$$\begin{array}{ccc} 10 & 40 & & 1 & 2 & 3 \\ 20 & 50 & & 4 & 5 & 6 \\ 30 & 60 & & 7 & 8 & 9 \\ & & & 10 & 11 & 12 \end{array} \quad \text{Illegal multiplication}$$

If we attempt to multiply these two matrices, we start (as always) with the first row of the first matrix, times the first column of the second matrix:

$$\begin{bmatrix} 10 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} . \text{ But this is an illegal multiplication; the items don't line up,}$$

since there are two elements in the row and four in the column. So you cannot multiply these two matrices.

This example illustrates two vital properties of matrix multiplication.

- The number of columns in the first matrix, and the number of rows in the second matrix, must be equal. Otherwise, you cannot perform the multiplication.
- Matrix multiplication is not **commutative**—which is a fancy way of saying, order matters. If you reverse the order of a matrix multiplication, you may get a different answer, or you may (as in this case) get no answer at all.

The Identity Matrix

This module introduces the identity matrix and its properties.

When multiplying numbers, the number 1 has a special property: when you multiply 1 by any number, you get that same number back. We can express this property as an algebraic generalization:

Equation:

$$1x = x$$

The matrix that has this property is referred to as the **identity matrix**.

Definition of Identity Matrix

The **identity matrix**, designated as $[I]$, is defined by the property:

$$[A][I] = [I][A] = [A]$$

Note that the definition of $[I]$ stipulates that the multiplication must **commute**—that is, it must yield the same answer no matter which order you multiply in. This is important because, for most matrices, multiplication does not commute.

What matrix has this property? Your first guess might be a matrix full of 1s, but that doesn't work:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$	$\text{so } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is not an identity matrix}$
--	---

The matrix that **does** work is a diagonal stretch of 1s, with all other elements being 0.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	<p>so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for 2x2 matrices</p>
$\begin{bmatrix} 2 & 5 & 9 & 1 & 0 & 0 \\ \pi & -2 & 8 & 0 & 1 & 0 \\ -3 & 1/2 & 8.3 & 0 & 0 & 1 \end{bmatrix} =$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity for 3x3 matrices

You should confirm those multiplications for yourself, and also confirm that they work in reverse order (as the definition requires).

Hence, we are led from the definition to:

The Identity Matrix

For any square matrix, its identity matrix is a diagonal stretch of 1s going from the upper-left-hand corner to the lower-right, with all other elements being 0. **Non-square** matrices do not have an identity. That is, for a non-square matrix $[A]$, there is no matrix such that $[A][I] = [I][A] = [A]$.

Why no identity for a non-square matrix? Because of the requirement of commutativity. For a non-square matrix $[A]$ you might be able to find a matrix $[I]$ such that $[A][I] = [A]$; however, if you reverse the order, you will be left with an illegal multiplication.

The Inverse Matrix

This module introduces the inverse matrix and its properties.

We have seen that the number 1 plays a special role in multiplication, because $1x = x$.

The **inverse of a number** is defined as the number that multiplies by that number to give 1: b is the inverse of a if $ab = 1$. Hence, the inverse of 3 is $\frac{1}{3}$; the inverse of $\frac{-5}{8} = \frac{-8}{5}$. Every number except 0 has an inverse.

By analogy, the **inverse of a matrix** multiplies by that matrix to give the identity matrix.

Definition of Inverse Matrix

The **inverse of matrix** $[A]$, designated as $[A]^{-1}$, is defined by the property:

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

The superscript -1 is being used here in a similar way to its use in functions. Recall that $f^{-1}(x)$ does not designate an exponent of any kind, but instead, an inverse function. In the same way, $[A]^{-1}$ does not denote an exponent, but an inverse **matrix**.

Note that, just as in the definition of the identity matrix, this definition requires commutativity—the multiplication must work the same in either order.

Note also that only square matrices can have an inverse. Why? The definition of an inverse matrix is **based on** the identity matrix $[I]$, and we already said that only square matrices even **have** an identity!

How do you **find** an inverse matrix? The method comes directly from the definition, with a little algebra.

Example: Finding an Inverse Matrix

Find the inverse of $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

The problem

Example: Finding an Inverse Matrix

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is the key step. It establishes $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get [I]. When we solve for the four variables a, b, c, and d, we will have found our inverse matrix.

Example: Finding an Inverse Matrix

$$\begin{bmatrix} 3a + 4c & 3b + 4d \\ 5a + 6c & 5b + 6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Do the multiplication. (You should check this step for yourself, it's great practice. For instance, you start by multiplying first row x first column, and you get $3a+4c$.)

$$3a + 4c = 1 \quad 3b + 4d = 0 \quad 5a + 6c = 0 \quad 5b + 6d = 1$$

Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

Example: Finding an Inverse Matrix

$$a = -3 \quad b = 2 \quad c = 2\frac{1}{2} \quad d = -1\frac{1}{2}$$

Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

So the inverse is: $\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$

Having found the four variables, we have found the inverse.

Did it work? Let's find out.

Testing our Inverse Matrix

$$\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$\stackrel{?}{=}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The definition of an inverse matrix: if we have indeed found an inverse, then when we multiply it by the original matrix, we should get [I].

Testing our Inverse Matrix

$$\begin{bmatrix} (-3)(3) + (2)(5) & (-3)(4) + (2)(6) \\ (2\frac{1}{2})(3) + (-1\frac{1}{2})(5) & (2\frac{1}{2})(4) + (-1\frac{1}{2})(6) \end{bmatrix}$$

$\stackrel{?}{=}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Do the multiplication.

$$\begin{bmatrix} -9 + 10 & -12 + 12 \\ 7\frac{1}{2} - 7\frac{1}{2} & 10 - 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



It works!

Note that, to fully test it, we would have to try the multiplication in **both orders**. Why? Because, in general, changing the order of a matrix multiplication changes the answer; but the definition of an inverse matrix specifies that it must work both ways! Only one order was shown above, so technically, we have only half-tested this inverse.

This process does not have to be memorized: it should make logical sense. Everything we have learned about matrices should make logical sense, except for the very arbitrary-looking definition of matrix multiplication.

Matrix Concepts -- Matrices on Calculators

Many modern graphing calculators have all the basic matrix operations built into them. The following is a brief overview of how to work with matrices on a TI-83, TI-83 Plus, TI-84, or TI-84 Plus.

The calculator has room to store up to ten matrices at once. It refers to these matrices as [A], [B], and so on, through [J]. Note that these are **not** the same as the 26 lettered memories used for numbers.

The following steps will walk you through the process of entering and manipulating matrices.

1. Hit the **MATRX** button. On a TI-83, this is a standalone button; on a TI-83 Plus, you first hit **2nd** and then **MATRIX** (above the x^{-1} button). The resulting display is a list of all the available matrices. (You have to scroll down if you want to see the ones below **[G]**.)

```
NAMES MATH EDIT
1: [A]
2: [B]
3: [C]
4: [D]
5: [E]
6: [F]
7: [G]
```

2. Hit the **right arrow key** ► twice, to move the focus from **NAMES** to **EDIT**. This signals that you want to create, or change, a matrix.

```
NAMES MATH EDIT
1: [A]
2: [B]
3: [C]
4: [D]
5: [E]
6: [F]
7: [G]
```

3. Hit the number **1** to indicate that you want to edit the **first** matrix, **[A]**.
4. Hit **4 ENTER 3 ENTER** to indicate that you want to create a 4x3 matrix. (4 rows, 3 columns.)

```
MATRIX[A] 4x3
[0 0 0 ]
[0 0 0 ]
[0 0 0 ]
[0 0 0 ]
```

```
1,1=0
```

5. Hit **1 ENTER 2 ENTER 3 ENTER 4 ENTER 5 ENTER 6 ENTER 7 ENTER 8 ENTER 9 ENTER 10 ENTER 11 ENTER 12 ENTER** This fills in the matrix with those numbers (you can watch it fill as you go). If you make a mistake, you can use the arrow keys to move around in the matrix until the screen looks like the picture below.

```
MATRIX[A] 4 x3
[[1 2 3]
 [4 5 6]
 [7 8 9]
 [10 11 12]]
```

4, 3=12

6. Hit **2nd Quit** to return to the main screen.
7. Return to the main matrix menu, as before. However, this time, do **not** hit the right arrow to go to the **EDIT** menu. Instead, from the NAMES menu, hit the number **1**. This puts **[A]** on the main screen. Then hit **ENTER** to display matrix **[A]**.

```
[A]
[[1 2 3]
 [4 5 6]
 [7 8 9]
 [10 11 12]]
```

8. Go through the process (steps 1-7) again, with a few changes. This time, define matrix **[B]** instead of matrix **[A]**. (This will change step 3: once you are in the **EDIT** menu, you will hit a **2** instead of a 1.) Define **[B]** as a 3x2 matrix in step 4. Then, in step 5, enter the following numbers:

Equation:

$$\begin{matrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{matrix}$$

When you are done, and have returned to the main screen and punched 2 in the NAMES menu (step 7), your main screen should look like this:

```
[B]
[[10 40]
 [20 50]
 [30 60]]
```

9. Now, type the following keys, watching the calculator as you do so. TI-83 Plus users should always remember to hit **2nd MATRIX** instead of just **MATRIX**. **MATRIX 1 + MATRIX 2**

```
[B]
[[10 40]
 [20 50]
 [30 60]]
[A]+[B]
```

This instructs the computer to add the two matrices. Now hit **ENTER**


```
ERR: DIM MISMATCH
1:Quit
2:Goto
```

Hey, what happened? You asked the computer to add two matrices. But these matrices have **different dimensions**. Remember that you can only add two matrices if they have the same dimensions—that is, the same number of rows as columns. So you got an “Error: Dimension Mismatch.” Hit **ENTER** to get out of this error and return to the main screen.

10. Now try the same sequence without the + key: **MATRIX 1 MATRIX 2 ENTER**

```
{A} {B}
      [20 50]
      [30 60]
{A} {B}
      [140 320 ]
      [320 770 ]
      [500 1220]
      [680 1670]
```

This instructs the calculator to **multiply** the two matrices. This is a legal multiplication—in fact, you may recognize it as the multiplication that we did earlier. The calculator displays the result that

we found by hand:

$$\begin{array}{ccc|cc} 1 & 2 & 3 & 10 & 40 \\ 4 & 5 & 6 & 20 & 50 \\ 7 & 8 & 9 & 30 & 60 \\ 10 & 11 & 12 & 680 & 1670 \end{array} = \begin{array}{cc} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{array}$$

11. Enter a third matrix, matrix $[C] = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. When you confirm that it is

entered correctly, the screen should look like this:

```
{C}
      [3 4]
      [5 6]
```

Now type **MATRIX 3 x-1 ENTER**

```
{C}
      [3 4]
      [5 6]
{C}^{-1}
      [[-3 2]
      [2.5 -1.5]]
```

This takes the **inverse** of matrix $[C]$. Note that the answer matches the inverse matrix that we found before.

12. Type **MATRIX 3 x-1 MATRIX 3 ENTER**

```
[C]-1      [5 6]
           [[-3  2]
           [2.5 -1.5]]
[C]-1[C]  [[1 0]
           [0 1]]
```

■
This instructs the calculator to multiply matrix [C]⁻¹ times matrix [C].
The answer, of course, is the 2×2 identity matrix [I].

Determinants

This module covers matrix determinants and their uses.

The Determinant of a 2x2 Matrix

In the exercise “Inverse of the Generic 2x2 Matrix,” you found that the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula can be used to very quickly find the inverse of any 2x2 matrix.

Note that if $ad-bc = 0$, the formula does not work, since it puts a 0 in the denominator. This tells us that, for any 2x2 matrix, if $ad-bc = 0$ the matrix **has no inverse**.

The quantity $ad-bc$ is therefore seen to have a special importance for 2x2 matrices, and it is accorded a special name: the “determinant.” Determinants are represented mathematically with absolute value signs: the determinant of matrix $[A]$ is $|A|$.

Definition of the Determinant of a 2x2 Matrix

If matrix $\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is the number $|A| = ad - bc$.

For instance, for the matrix $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, the determinant is $(3)(6)-(4)(5) = -2$.

Note that the determinant is a **number**, not a matrix. It is a special number that is **associated with** a matrix.

We said earlier that “if $ad-bc = 0$ the matrix **has no inverse**.” We can now restate this result.

Any square matrix whose determinant is **not** 0, has an inverse matrix. Any square matrix with determinant 0 has no inverse.

This very important result is analogous to the result stated earlier for numbers: every number except 0 has an inverse.

The Determinant of a 3x3 Matrix (or larger)

Any **square matrix** has a determinant—an important number associated with that matrix. Non-square matrices do not have a determinant.

How do you find the determinant of a 3x3 matrix? The method presented here is referred to as “expansion by minors.” There are other methods, but they turn out to be mathematically equivalent to this one: that is, they end up doing the same arithmetic and arriving at the same answer.

Example: Finding the Determinant of a 3x3 Matrix	
Find the determinant $\begin{vmatrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix}$	The problem.
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.	We’re going to walk through the top row, one element at a time, starting with the first element (the 2). In each case, begin by crossing out the row and column that contain that number.
$\begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} = (8)(1) - (3)(1) = 5$	Once you cross out one row and column, you are left with a 2x2 matrix (a “minor”). Take the determinant of that matrix.
$2(5) = 10$	Now, that “minor” is what we got by crossing out a 2 in the top row. Multiply that number in the top row (2) by the determinant of the minor (5).

Example: Finding the Determinant of a 3x3 Matrix	
<p>***SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.*** $(10)(1) - (3)(1) = 74(7) = 28$</p>	<p>Same operation for the second element in the row (the 4 in this case)...</p>
<p>***SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.*** $(10)(1) - (8)(1) = 25(2) = 10$</p>	<p>...and the third (the 5 in this case).</p>
<p>$+10 - 28 + 10 = -8$</p>	<p>Take these numbers, and alternately add and subtract them; add the first, subtract the second, add the third. The result of all that is the determinant.</p>

This entire process can be written more concisely as:

$$\begin{vmatrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} - 4 \begin{vmatrix} 10 & 3 \\ 1 & 1 \end{vmatrix} + 5 \begin{vmatrix} 10 & 8 \\ 1 & 1 \end{vmatrix} = 2(5 - 4) - 4(10 - 3) + 5(10 - 8) = 2(1) - 4(7) + 5(2) = -8$$

This method of “expansion of minors” can be extended upward to any higher-order square matrix. For instance, for a 4x4 matrix, each “minor” that is left when you cross out a row and column is a 3x3 matrix. To find the determinant of the 4x4, you have to find the determinants of all four 3x3 minors!

Fortunately, your calculator can also find determinants. Enter the matrix given above as matrix [D]. Then type:

MATRX ► 1MATRX 4) ENTER

The screen should now look like this:

```

[0]      [[2  4  5]
          [10 8  3]
          [1  1  1]]
det([0])      -8
■

```

If you watched the calculator during that sequence, you saw that the right-arrow key took you to the **MATH** submenu within the **MATRIX** menus. The first item in that submenu is **DET** (which means “determinant of.”

What does the determinant mean? It turns out that this particular odd set of operations has a surprising number of applications. We have already seen one—in the case of a 2x2 matrix, the determinant is part of the inverse. And for any square matrix, the determinant tells you whether the matrix **has** an inverse at all.

Another application is for finding the area of triangles. To find the area of a triangle whose vertices are (a,b), (c,d), and (e,f), you can use the formula: Area =

$$\frac{1}{2} \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}.$$

Hence, if you draw a triangle with vertices (2,10), (4,8), and (5,3), the above calculation shows that the area of this triangle will be 4.

Solving Linear Equations

This module explains how to use matrices to solve linear equations.

At this point, you may be left with a pretty negative feeling about matrices. The initial few ideas—adding matrices, subtracting them, multiplying a matrix by a constant, and matrix equality—seem almost too obvious to be worth talking about. On the other hand, multiplying matrices and taking determinants seem to be strange, arbitrary sequences of steps with little or no purpose.

A great deal of it comes together in solving linear equations. We have seen, in the chapter on simultaneous equations, how to solve two equations with two unknowns. But suppose we have three equations with three unknowns? Or four, or five? Such situations are more common than you might suppose in the real world. And even if you are allowed to use a calculator, it is not at all obvious how to solve such a problem in a reasonable amount of time.

Surprisingly, the things we have learned about matrix multiplication, about the identity matrix, about inverse matrices, and about matrix equality, give us a very fast way to solve such problems on a calculator!

Consider the following example, three equations with three unknowns:

Equation:

$$x + 2y - z = 11$$

Equation:

$$2x - y + 3z = 7$$

Equation:

$$7x - 3y - 2z = 2$$

Define a 3×3 matrix $[A]$ which is the coefficients of all the variables on the left side of the equal signs:

$$[A] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix}$$

Define a 3×1 matrix $[B]$ which is the numbers on the **right** side of the equal signs:

$$[B] = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$$

Punch these matrices into your calculator, and then ask the calculator for $[A^{-1}][B]$: that is, the inverse of matrix $[A]$, multiplied by matrix $[B]$.

$$[A]^{-1}[B] = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

The calculator responds with a 3×1 matrix which is **all three answers**. In this case, $x = 3$, $y = 5$, and $z = 2$.

The whole process takes no longer than it takes to punch a few matrices into the calculator. And it works just as quickly for 4 equations with 4 unknowns, or 5, **etc.**

Huh? Why the heck did that work?

Solving linear equations in this way is fast and easy. But with just a little work—and with the formalisms that we have developed so far about matrices—we can also show why this method works.

Step 1: In Which We Replace Three Linear Equations With One Matrix Equation

First of all, consider the following matrix equation:

$$\begin{array}{rcl} x + 2y - z & & 11 \\ 2x - y + 3z & = & 7 \\ 7x - 3y - 2z & & 2 \end{array}$$

The matrix on the left may **look** like a 3×3 matrix, but it is actually a 3×1 matrix. The top element is $x + 2y - z$ (all one big number), and so on.

Remember what it means for two matrices to be equal to each other. They have to have the same dimensions (



). And all **the elements have to be equal to each other**. So for this matrix equation to be true, all three of the following equations must be satisfied:

Equation:

$$x + 2y - z = 11$$

Equation:

$$2x - y + 3z = 7$$

Equation:

$$7x - 3y - 2z = 2$$

Look familiar? Hey, this is the three equations we started with! The point is that this **one matrix equation** is equivalent to those **three linear equations**. We can replace the original three equations with one matrix equation, and then set out to solve that.

Step 2: In Which We Replace a Simple Matrix Equation with a More Complicated One

Do the following matrix multiplication. (You will need to do this by hand—since it has variables, your calculator can't do it for you.)

$$\begin{array}{cccc} 1 & 2 & -1 & x \\ 2 & -1 & 3 & y \\ 7 & -3 & -2 & z \end{array}$$

If you did it correctly, you should have wound up with the following 3×1 matrix:

Equation:

$$\begin{array}{l} x + 2y - z \\ 2x - y + 3z \\ 7x - 3y - 2z \end{array}$$

Once again, we pause to say...hey, that looks familiar! Yes, it's the matrix that we used in Step 1. So we can now rewrite the **matrix equation** from Step 1 in this way:

$$\begin{array}{cccccc} 1 & 2 & -1 & x & & 11 \\ 2 & -1 & 3 & y & = & 7 \\ 7 & -3 & -2 & z & & 2 \end{array}$$

Stop for a moment and make sure you're following all this. I have shown, in two separate steps, that this **matrix equation** is equivalent to the three **linear equations** that we started with.

But this matrix equation has a nice property that the previous one did not. The first matrix (which we called $[A]$ a long time ago) and the third one ($[B]$) contain only numbers. If we refer to the middle matrix as $[X]$ then we can write our equation more concisely:

$$[A][X] = [B], \text{ where } [A] = \begin{array}{ccc} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{array}, [X] = \begin{array}{c} x \\ y \\ z \end{array}, \text{ and } [B] = \begin{array}{c} 11 \\ 7 \\ 2 \end{array}$$

Most importantly, $[X]$ contains the three variables we want to solve for! If we can solve this equation for $[X]$ we will have found our three variables x ,

y , and z .

Step 3: In Which We Solve a Matrix Equation

We have rewritten our original equations as $[A][X] = [B]$, and redefined our original goal as “solve this matrix equation for $[X]$.” If these were numbers, we would divide both sides by $[A]$. But these are matrices, and we have never defined a division operation for matrices. Fortunately, we can do something just as good, which is **multiplying both sides** by $[A]^{-1}$. (Just as, with numbers, you can replace “dividing by 3” with “multiplying by $\frac{1}{3}$.”)

Solving a Matrix Equation	
$[A][X] = [B]$	The problem.
$[A]^{-1}[A][X] = [A]^{-1}[B]$	Multiply both sides by $[A]^{-1}$, on the left. (Remember order matters! If we multiplied by $[A]^{-1}$ on the right, that would be doing something different.)
$[I][X] = [A]^{-1}[B]$	$[A]^{-1}[A] = [I]$ by the definition of an inverse matrix.
$[X] = [A]^{-1}[B]$	$[I]$ times anything is itself, by definition of the identity matrix.

So we're done! $[X]$, which contains exactly the variables we are looking for, has been shown to be $[A]^{-1}[B]$. This is why we can punch that formula into our calculator and find the answers instantly.

Let's try one more example

Equation:

$$5x - 3y - 2z = 4$$

Equation:

$$x + y - 7z = 7$$

Equation:

$$10x - 6y - 4z = 10$$

We don't have to derive the formula again—we can just use it. Enter the following into your calculator:

$$[A] = \begin{bmatrix} 5 & -3 & -2 \\ 1 & 1 & -7 \\ 10 & -6 & -4 \end{bmatrix} \quad [B] = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

Then ask the calculator for $[A]^{-1}[B]$.

```
[[5 -3 -2]
 [1 1 -7]
 [10 -6 -4]]
[B]
[[4 ]
 [7 ]
 [10]]
[A]^-1[B]
```

The result?

```
ERR: SINGULAR MAT
1:Quit
2:Goto
```

What happened? To understand this error, try the following:

Hit **ENTER** to get out of the error, and then hit **<MATRX>** ► **1 <MATRX>**
1) ENTER

```
[B]
      [[4 ]
      [7 ]
      [10]]
[A]⁻¹[B]
det<[A]> 0
■
```

Aha! Matrix $[A]$ has a determinant of 0. **A matrix with 0 determinant has no inverse.** So the operation you asked the calculator for, $[A]^{-1}[B]$, is impossible.

What does this tell us about our original equations? They have no solution. To see why this is so, double the first equation and compare it with the third—it should become apparent that both equations cannot be true at the same time.

Data Concepts -- Introduction

Conceptual Explanations: Modeling Data with Functions

In school, you generally start with a function and work from there to numbers. “Newton’s Law tells us that $F = ma$. So if you push on a 3kg object with a 12N force, what will the acceleration be?”

In real life, the work often goes the other way. Newton didn’t start out **knowing** that $F = ma$; he observed the world around him, and **concluded** that $F = ma$. In science, you begin with data—that is, numbers—and attempt to find a mathematical function that will **model** the data. Then you use that function to make predictions for new data. If the predictions come true, you gain confidence in your model.

So, this unit is about a few processes that can be used to look at a set of numbers and find a function that relates them.

Direct and Inverse Variations

This module introduces the concept of direct and inverse variation in Algebra in preparation for modeling data with functions.

Direct Variation

As a simple example, consider the variable c which is the number of cars in a parking lot, and the variable t which is the number of tires in the parking lot. Assuming each car has four tires, we might see numbers like this.

c (number of cars)	t (number of tires)
0	0
1	4
2	8
3	12
4	16

These two columns stand in a very particular relationship to each other which is referred to as **direct variation**.

Definition of “Direct Variation”

Two variables are in “direct variation” with each other if the following relationship holds: whenever one variable doubles, the other variable doubles. Whenever one variable triples, the other variable triples. And so on.

When the left-hand column goes up, the right-hand column goes up. This is characteristic of direct variation, but it does not **prove** a direct variation. The function $y = x + 1$ has the characteristic that whenever x goes up, y also goes up; however, it does not fulfill the definition of direct variation.

The equation for this particular function is, of course, $t(c) = 4c$. In general, direct variation always takes the form $y = kx$, where k is some constant—a number, not a function of x . This number is referred to as the **constant of variation**.

Note that, in real life, these relationships are not always exact! For instance, suppose m is the number of men in the room, and w is the **weight** of all the men in the room. The data might appear something like this:

m (number of men)	w (total weight of men, in pounds)
0	0
1	160
2	330
3	475
4	655

Not all men weigh the same. So this is not **exactly** a direct variation. However, looking at these numbers, you would have a very good reason to suspect that the relationship is more or less direct variation.

How can you confirm this? Recall that if this is direct variation, then it follows the equation $w = km$, or $w/m = k$. So for direct variation, we

would expect the ratio w/m to be approximately the same in every case. If you compute this ratio for every pair of numbers in the above table, you will see that it does indeed come out **approximately** the same in each case. (Try it!) So this is a good **candidate** for direct variation.

Inverse Variation

Suppose 5 cars all travel 120 miles. These cars get different mileage. How much gas does each one use? Let m be the miles per gallon that a car gets, and g be the number of gallons of gas it uses. Then the table might look something like this.

m (miles/gallon)	g (gallons of gas used to travel 120 miles)
10	12
20	6
30	4
40	3
60	2

These variables display an **inverse relationship**.

Definition of “Inverse Variation”

Two variables are in “inverse variation” with each other if the following relationship holds: whenever one variable doubles, the other variable halves. Whenever one variable triples, the other variable drops in a third. And so on.

Note that as the first column gets **bigger**, the second column gets **smaller**. This is suggestive of an inverse relationship, but it is not a guarantee. $y = 10 - x$ would also have that property, and it is not inverse variation.

The equation for this particular function is $g = 120/m$. In general, inverse variation can always be expressed as $y = k/x$, where k is once again the **constant of variation**.

If $y = k/x$, then of course $xy = k$. So inverse variation has the characteristic that when you **multiply** the two variables, you get a constant. In this example, you will always get 120. With real life data, you may not always get **exactly** the same answer; but if you always get **approximately** the same answer, that is a good indication of an inverse relationship.

More Complex Examples

In the year 1600, Johannes Kepler sat down with the data that his teacher, Tycho Brahe, had collected after decades of carefully observing the planets. Among Brahe's data was the **period** of each planet's orbit (how many years it takes to go around the sun), and the **semimajor axis** of the orbit (which is sort of like a radius, but not quite—more on this in “Ellipses”). Today, these figures look something like this.

Planet	Semimajor Axis a (1010 meters)	Period T (years)
Mercury	5.79	0.241
Venus	10.8	0.615
Earth	15.0	1 (*duh)

Planet	Semimajor Axis a (1010 meters)	Period T (years)
Mars	22.8	1.88
Jupiter	77.8	11.9
Saturn	143	29.5
Uranus	287	84
Neptune	450	165

What can we make of this data? As a goes up, T clearly also goes up. But they are not directly proportional. For instance, looking at the numbers for Uranus and Neptune, we see that 165 is almost exactly twice 84; but 450 is much less than twice 287. Is there a consistent pattern? Kepler went down in history for figuring out that **the square of the period is directly proportional to the cube of the semimajor axis**: in numbers, $T^2 = ka^3$. You can confirm this for yourself, using the numbers above. (What is k ?)

So we see that the concepts of “directly proportional” and “inversely proportional” can be applied to situations more complex than $y = kx$ or $y = k/x$.

The situation becomes more interesting still when **multiple independent variables** are involved. For instance, Isaac Newton was able to explain Kepler’s results by proposing that every body in the world exerts a gravitational field that obeys the following two laws.

- When the mass of the body doubles, the strength of the gravitational field doubles
- When the distance from the body doubles, the strength of the field drops in a fourth

Science texts express these laws more concisely: the field is **directly proportional to the mass, and inversely proportional to the square of**

the radius. It may seem as if these two statements require two different equations. But instead, they are two different clues to finding the **one** equation that allows you to find the gravitational field F at a distance r from a given mass m . That one equation is $F = \frac{Gm}{r^2}$ where G , the constant of proportionality, is one of the universal constants of nature. This does not come from **combining** the two equations $F = km$ and $F = \frac{k}{r^2}$ as a composite function or anything else. Rather, it is one equation that expresses both relationships properly: doubling the mass doubles the field, and doubling the radius drops the field in a fourth.

Data Concepts -- Linear Functions

Finding a Linear Function for any Two Points

In an earlier unit, we did a great deal of work with the equation for the **height** of a ball thrown straight up into the air. Now, suppose you want an equation for the **speed** of such a ball. Not knowing the correct formula, you run an experiment, and you measure the two data points.

t (time)	v (velocity, or speed)
1 second	50 ft/sec
3 seconds	18 ft/sec

Obviously, the ball is slowing down as it travels upward. Based on these two data points, what function $v(t)$ might model the speed of the ball?

Given any two points, the simplest equation is always a line. We have two points, (1,50) and (3,18). How do we find the equation for that line? Recall that every line can be written in the form:

Equation:

$$y = mx + b$$

If we can find the m and b for our particular line, we will have the formula.

Here is the key: **if our line contains the point (1,50) that means that when we plug in the x-value 1, we must get the y-value 50.**

The equation contains the point (1, 50). That tells us that...

$$y = mx + b$$

When this is 1, so... $50 = m(1) + b$
this must be 50

Similarly, we can use the point (3,18) to generate the equation $18 = m(3) + b$. So now, in order to find m and b , we simply have to solve two equations and two unknowns! We can solve them either by substitution or elimination: the example below uses substitution.

Equation:

$$m + b = 50 \rightarrow b = 50 - m$$

Equation:

$$3m + b = 18 \rightarrow 3m + (50 - m) = 18$$

Equation:

$$2m + 50 = 18$$

Equation:

$$2m = -32$$

Equation:

$$m = -16$$

Equation:

$$b = 50 - (-16) = 66$$

So we have found m and b . Since these are the unknowns in the equation $y = mx + b$, the equation we are looking for is:

Equation:

$$y = -16x + 66$$

Based on this equation, we would expect, for instance, that after 4 seconds, the speed would be 2 ft/sec. If we measured the speed after 4 seconds and found this result, we would gain confidence that our formula is correct.

Finding a Parabolic Function for any Three Points

This module introduces the concept of modeling data with parabolic functions in Algebra.

Finding a Parabolic Function for any Three Points

Any two points are joined by a line. Any **three** points are joined by a **vertical parabola**.

Let's start once again with the exceptions. Once again, if any two of the points are vertically aligned, then no function can join them. However, there is no additional exception—if all three points **lie on a line**, then no parabola joins them. For instance, no parabola contains the three points (1,3), (2,5), and (5,11). In real life, of course, if we wanted to model those three points, we would be perfectly happy to use the line $y = 2x + 1$ instead of a parabola.

However, if three points **are not vertically aligned** and **do not lie on a line**, it is always possible to find a vertical parabola that joins them. The process is very similar to the process we used for a line, except that the starting equation is different.

Example:

Finding a Vertical Parabola to Fit Three Points

Find a vertical parabola containing the points (-2,5), (-1,6), and (3,-10).

The problem. As with our example earlier, this problem could easily come from an attempt to find a function to model real-world data.

$$y = ax^2 + bx + c$$

This is the equation for any vertical parabola. Our job is to find a , b , and c . Note that this starting point is the same for any problem with three points, just as any problem with two points starts out $y = mx + b$.

$$5 = a(-2)^2 + b(-2) + c$$

$$6 = a(-1)^2 + b(-1) + c$$

$$-10 = a(3)^2 + b(3) + c$$

Each point represents an (x, y) pair that must create a true equation in our function. Hence, we can plug each point in for x and y to find three equations that must be true. We can now solve for our 3 unknowns. Rewrite the above three equations in a more standard form:

$$4a - 2b + c = 5$$

$$a - b + c = 6$$

$$9a + 3b + c = -10$$

Uh-oh. Now what? In the linear example, we used elimination or substitution to solve for the two variables. How do we solve three? Oh, yeah. Matrices! Rewrite the above three equations as $[A][X] = [B]$, where

$$[X] = \begin{matrix} a \\ b \\ c \end{matrix} \text{ is what we want.}$$

$$[A] = \begin{matrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 9 & 3 & 1 \end{matrix}$$

$$[B] = \begin{matrix} 5 \\ 6 \\ -10 \end{matrix}$$

$$[A]^{-1} [B] = \begin{matrix} -1 \\ -2 \\ 5 \end{matrix}$$

From the calculator, of course. Remember what this means! It means that $a = -1$, $b = -2$, and $c = 5$. We can now plug these into our original equation, $y = ax^2 + bx + c$.

$$y = -x^2 - 2x + 5$$

So this is the equation we were looking for.

Did it work? Remember that we were looking for a parabola that contained the three points $(-2,5)$, $(-1,6)$, and $(3,-10)$. If this parabola contains those three points, then our job is done. Let's try the first point.

Equation:

$$5 \stackrel{?}{=} -(-2)^2 - 2(-2) + 5$$

$$= -4 + 4 + 5$$



So the parabola does contain the point $(-2,5)$. You can confirm for yourself that it also contains the other two points.

Finally, remember what this means! If we had measured some real-world phenomenon and found the three points $(-2,5)$, $(-1,6)$, and $(3,-10)$, we would now **suspect** that the function $y = -x^2 - 2x + 5$ might serve as a model for this phenomenon.

This model **predicts** that if we make a measurement at $x = -3$ we will find that $y = 2$. If we made such a measurement and it matched the prediction, we would gain greater confidence in our model. On the other hand, if the measurement was far off the prediction, we would have to rethink our model.

A surprising application: “secret sharing”

Bank vaults are commonly secured by a method called “secret sharing.” The goal of a secret sharing system runs something like this: if any three employees enter their secret codes at the **same time**, the vault will open. But any two employees together cannot open the vault.

Secret sharing is implemented as follows.

- Choose a parabolic function—that is, choose the numbers a , b , and c in the equation $y = ax^2 + bx + c$. This function is chosen at random, and is not programmed into the vault or given to any employee.
- The actual number that will open the vault is the y-intercept of the parabola: that is, the y-value of the parabola when $x = 0$. This number is not given to any employee.
- Each employee’s secret code is **one point on the parabola**.

When three employees enter their secret codes at the same time, the vault computer uses the three points to compute a , b , and c for the parabola. As we have seen, this computation can be done quickly and easy using inverse matrices and matrix multiplication, both of which are easy algorithms to program into a computer. Once the computer has those three numbers, it computes the y-value when $x = 0$, and uses this number to open the vault.

Any three employees—that is, any three points—are enough to uniquely specify the parabola. But if you only have two points, you are no closer to the answer than when you started: the secret y value could still be, literally, any number at all.

Note also that the system is easily extendable. That is, if you want to say that **four** employees are required to open the vault, you just move up to a third-order polynomial, $y = ax^3 + bx^2 + c + d$. The resulting equations—four equations with four unknowns—are just as easy, with matrices, as three were.

Data Concepts -- Regression on the the Calculator

Regression on the Calculator

What Kepler did is an example of “regression”: finding an equation that models a particular set of data.

Kepler became famous because regression is hard. Who would have thought to look for $T^2 = ka^3$? Especially when you consider all the other equations that would still have a “this goes up, that goes up” relationship, such as $T = 2^a + 7$, or $T = \frac{1}{10-a}$, or maybe $T = 3\log_5 a$?

Fortunately, we have a tool that Kepler did not have: the modern computer. Mathematical programs and graphing calculators can take a set of points, and find the line or curve of “best fit” to model the data.

As an example of this process, suppose that you have run an experiment and generated three data points: (2,1), (4,3), and (5,8). What function might model those data?

Note:Note: the directions below are given for a TI-83 or compatible calculator. Many other calculators can perform the same functions, but the implementation details may look quite different.

Entering the data

1. Hit **STAT** to go into the Statistics menu.
2. Choose **Edit...** This brings you to a screen where you enter a bunch of L1 and L2 values.

FIXME: A LIST CAN NOT BE A TABLE ENTRY. Enter the L1 and L2 values as follows: for each data point, the x-coordinate is in the L1 list, and the y-coordinate is in the L2 list. The screen to the right shows the points (2,1), (4,3), and (5,8).



1. Hit **2nd QUIT** return to the main screen.

Viewing the data, and guessing at the shape

Once you have entered your points into the L1 and L2 lists, your calculator can show you a “scatterplot”—which is a pointlessly fancy word for “a graph of a bunch of points,” like you used to make when you were first learning what graphing was.

1. Hit **WINDOW** (near the upper-left-hand corner of the calculator).
2. Set the variables Xmin, Xmax, Ymin, and Ymax appropriately. For instance, for the three points shown above, the x-values are 2, 4, and 5, and the y-values are 1, 3, and 8. So it might make sense to set Xmin=0, XMax=10, Ymin=0, and Ymax=10—and that is how I did it in the drawing below. Of course, there are many other settings you could use. But if you go through the whole process and don’t see any points, it’s a reasonable guess that your window is not set properly.
3. Hit **Y=** (upper-left-hand corner of the calculator).
4. Then, hit the up-arrow key, so the focus moves to **Plot1** (which will start blinking).
5. Hit **ENTER**. It’s actually impossible, at this point, to see that anything has happened. But if you down-arrow **away** from **Plot1**, you should see that it remains darkened (white letters on a black background, instead of the other way around). This indicates that it has been selected.

FIXME: A LIST CAN NOT BE A TABLE ENTRY. Hit **GRAPH** (upper-right-hand corner of the calculator). The calculator now displays the points. From the image, you can see that a quadratic (parabolic) or exponential function might be a reasonable guess, whereas a line or logarithmic function would be unlikely to fit.

At this point, looking at the data, it is often useful to categorize it in two ways.

First: is it **increasing or decreasing**? In our example, of course, the points are increasing. (Some data, of course, may be doing both at different times: consider, for instance, a parabola.)

Now, in the case of an increasing function, you can categorize it as one of the following.

	If it is increasing steadily, that suggests a line. (Remember that what makes a linear function linear is that it always goes up at the same rate, or slope!)
	If it is increasing more and more slowly, that suggests a logarithmic function. (A square root would also have this basic shape, but you cannot do a square-root-regression.)
	If it is increasing more and more quickly, that suggests an exponential function, or possibly the right side of a quadratic function (a parabola).

Decreasing functions can be categorized similarly, of course. If a function decreases and **then** increases, a parabola is probably the best fit. Functions

that go up, then down, then up again, are most likely to be higher-order polynomials.

Finding the formula

Once you have decided on the right shape, the hard work is done: the calculator takes care of the rest.

1. Hit **STAT** to return again to the Statistics menu.
2. Hit the right-arrow key to go to **CALC**.
3. At this point, you have several choices. **LinReg** will give you a line that best fits your points. **QuadReg** will give you a quadratic function, **aka** a second-order polynomial. There are also options for "cubic" (third-order polynomial), "quartic" (fourth-order polynomial), logarithmic, or exponential curves. Choose the one you want and hit **ENTER**.

The calculator does **not** graph your curve for you, but it does tell you what the curve is. For instance, if I run a **QuadReg** on the data above, the calculator gives me:

```
QuadReg
y=ax2+bx+c
a=1.333333333
b=-7
c=9.666666667
```

This tells me that the best quadratic fit for my data is the curve $y = 1.33x^2 - 7x + 9.67$. One way to double-check this, of course, is to enter $Y1 = 1.33x^2 - 7x + 9.67$ and then graph it, and see how closely it approximates the points!

Note: Remember that whatever option you choose, it will operate on the points you have entered in the L1 and L2 lists, so make sure your data is correctly entered there!

Starting over

There's just one more thing you have to know: once you've done this once, how do you clear out the lists to enter new ones? Here is one way to do it.

1. Hit **MEM** (you do this by hitting the yellow **2nd** key, and then hitting **+**).
2. This brings up a menu. Choose **ClrAllLists**.
3. Then—after you return to the main screen—hit **ENTER** and the lists are emptied out.

Conic Concepts

This module introduces the concept of conic sections in Algebra.

So far, we have talked about how to graph two shapes: lines, and parabolas. This unit will discuss parabolas in more depth. It will also discuss circles, ellipses, and hyperbolas. These shapes make up the group called the **conic sections**: all the shapes that can be created by intersecting a **plane** with a **double cone**.

On the left is a double cone . If you intersect the double cone with a horizontal plane, you get a circle . If you tilt the plane a bit, you get an ellipse (as in the bad clip art picture on the right). If you tilt the plane more, so it never hits the other side of the cone, you get a parabola . If the plane is vertical, so it hits both cones, you get a hyperbola .
--

We are going to discuss each of these shapes in some detail. Specifically, for each shape, we are going to provide...

- A formal **definition** of the shape, and
- The **formula** for graphing the shape

These two things—the definition, and the formula—may in many cases seem unrelated. But you will be doing work in the text exercises to show, for each shape, how the definition **leads to** the formula.

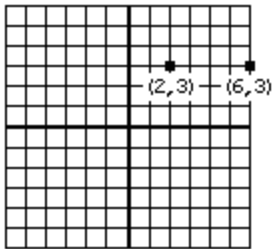
Conic Concepts -- Distance

The key mathematical formula for discussing all the shapes above is the **distance between two points**.

Many students are taught, at some point, the “distance formula” as a magic (and very strange-looking) rule. In fact, the distance formula comes directly from a bit of intuition...and the Pythagorean Theorem.

The intuition comes in finding the distance between two points that have one coordinate in common.

The distance between two points that have one coordinate in common



The drawing shows the points (2,3) and (6,3). Finding the distance between these points is easy: just count! Take your pen and move it along the paper, starting at (2,3) and moving to the right. Let's see...one unit gets you over to (3,3); the next unit gets you to (4,3)...a couple more units to (6,3). The distance from (2,3) to (6,3) is 4.

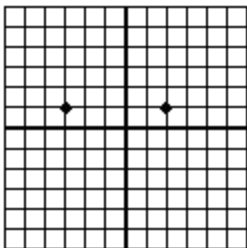
Of course, it would be tedious to count our way from (2,3) to (100,3). But we don't have to—in fact, you may have already guessed the faster way—we subtract the x coordinates.

- The distance from (2,3) to (6,3) is $6 - 2 = 4$
- The distance from (2,3) to (100,3) is $100 - 2 = 98$

And so on. We can write this generalization in words:

Note: Whenever two points lie on a horizontal line, you can find the distance between them by subtracting their x -coordinates.

This may seem pretty obvious in the examples given above. It's a little less obvious, but still true, if one of the x coordinates is negative.



The drawing above shows the numbers $(-3,1)$ and $(2,1)$. You can see that the distance between them is 5 (again, by counting). Does our generalization still work? Yes it does, because **subtracting a negative number** is the same as **adding a positive one**.

The distance from $(-3,1)$ to $(2,1)$ is $2 - (-3) = 5$

How can we express this generalization mathematically? If two points lie on a horizontal line, they have two different x -coordinates: call them x_1 and x_2 . But they have the **same** y -coordinate, so just call that y . So we can rewrite our generalization like this: “the distance between the points (x_1, y) and (x_2, y) is $x_2 - x_1$.” In our most recent example, $x_1 = -3$, $x_2 = 2$, and $y = 1$. So the generalization says “the distance between the points $(-3,1)$ and $(2,1)$ is $2 - (-3)$ ”, or 5.

But there's one problem left: what if we had chosen x_2 and x_1 the other way? Then the generalization would say “the distance between the points $(2,1)$ and $(-3,1)$ is $(-3) - 2$ ”, or -5. That isn't quite right: distances can never be negative. We get around this problem by taking the **absolute value** of the answer. This guarantees that, no matter what order the points are listed in, the distance will come out positive. So now we are ready for the correct mathematical generalization:

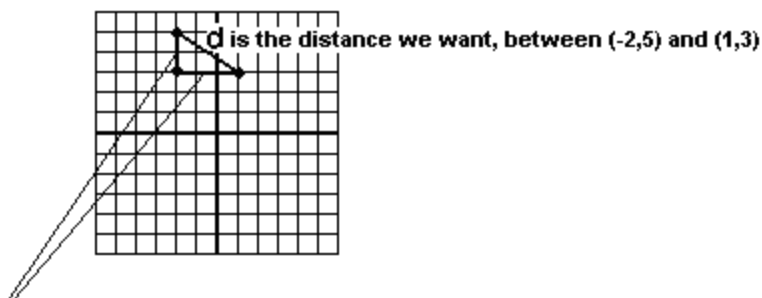
Distance Between Two Points on a Horizontal Line

The distance between the points (x_1, y) and (x_2, y) is $|x_2 - x_1|$

You may want to check this generalization with a few specific examples—try both negative and positive values of x_1 and x_2 . Then, to really test your understanding, write and test a similar generalization for two points that lie on a **vertical** line together. Both of these results will be needed for the more general case below.

The distance between two points that have no coordinate in common

So, what if two points have both coordinates different? As an example, consider the distance from $(-2, 5)$ to $(1, 3)$.



These two distances are not the distance we want, but they are easy to find. And they are related to the distance we want by the Pythagorean Theorem

The drawing shows these two points. The (diagonal) line between them has been labeled d : it is this line that we want the length of, since this line represents the distance between our two points.

The drawing also introduces a third point into the picture, the point $(-2, 3)$. The three points define the vertices of a right triangle. Based on our earlier discussion, you can see that the **vertical** line in this triangle is length $|5 - 3| = 2$. The **horizontal** line is length $|1 - (-2)| = 3$.

But it is the **diagonal** line that we want. And we can find that by using the Pythagorean Theorem, which tells us that $d^2 = 2^2 + 3^2$. So $d = \sqrt{13}$

If you repeat this process with the generic points (x_1, y_1) and (x_2, y_2) you arrive at the distance formula:

Distance between any two points

If d is the distance between the points (x_1, y_1) and (x_2, y_2) , then

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$x_2 - x_1$ is the horizontal distance, based on our earlier calculation. $y_2 - y_1$ is the vertical distance, and the entire formula is simply the Pythagorean Theorem restated in terms of coordinates.

And what about those absolute values we had to put in before? They were used to avoid negative distances. Since the distances in the above formulae are being **squared**, we no longer need the absolute values to insure that all answers will come out positive.

Circles

This module introduces the definition and formula of a circle, including example problems.

The Definition of a Circle

You've known all your life what a circle looks like. You probably know how to find the area and the circumference of a circle, given its radius. But what is the exact mathematical **definition** of a circle? Before you read the answer, you may want to think about the question for a minute. Try to think of a precise, specific definition of exactly what a circle is.

Below is the definition mathematicians use.

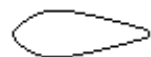
Definition of a Circle

The set of all points in a plane that are the **same distance from a given point** forms a **circle**. The point is known as the center of the **circle**, and the distance is known as the **radius**.

Mathematicians often seem to be deliberately obscuring things by creating complicated definitions for things you already understood anyway. But if you try to find a simpler definition of exactly what a circle is, you will be surprised at how difficult it is. Most people start with something like “a shape that is round all the way around.” That does describe a circle, but it also describes many other shapes, such as this pretzel:



So you start adding caveats like “it can't cross itself” and “it can't have any loose ends.” And then somebody draws an egg shape that fits all your criteria, and yet is still not a circle:



So you try to modify your definition further to exclude **that...** and by that time, the mathematician's definition is starting to look beautifully simple.

But does that original definition actually produce a circle? The following experiment is one of the best ways to convince yourself that it does.

Experiment: Drawing the Perfect Circle

1. Lay a piece of cardboard on the floor.
2. Thumbtack one end of a string to the cardboard.
3. Tie the other end of the string to your pen.
4. Pull the string as tight as you can, and then put the pen on the cardboard.
5. Pull the pen all the way around the thumbtack, keeping the string taut at all times.

The pen will touch every point on the cardboard that is **exactly one string-length away from the thumbtack**. And the resulting shape will be a circle. The cardboard is the **plane** in our definition, the thumbtack is the **center**, and the string length is the **radius**.

The purpose of this experiment is to convince yourself that if you take all the points in a plane that are a given distance from a given point, the result is a circle. We'll come back to this definition shortly, to clarify it and to show how it connects to the **mathematical formula** for a circle.

The Mathematical Formula for a Circle

You already know the formula for a line: $y = mx + b$. You know that m is the slope, and b is the y-intercept. Knowing all this, you can easily answer questions such as: “Draw the graph of $y = 2x - 3$ ” or “Find the equation of a line that contains the points (3,5) and (4,4).” If you are given the equation $3x + 2y = 6$, you know how to graph it in two steps: first **put it in the standard $y = mx + b$ form**, and then graph it.

All the conic sections are graphed in a similar way. There is a **standard form** which is very easy to graph, once you understand what all the parts mean. If you are given an equation that is not in standard form, you put it into the standard form, and then graph it.

So, to understand the formula below, think of it as the $y = mx + b$ of circles.

Mathematical Formula for a Circle

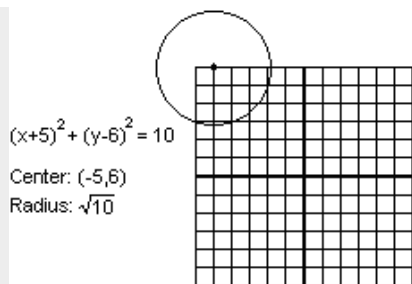
$(x-h)^2 + (y-k)^2 = r^2$ is a circle with center (h,k) and radius r

From this, it is very easy to graph a circle in standard form.

Example:

Graphing a Circle in Standard Form

<p>Graph $(x + 5)^2 + (y - 6)^2 = 10$</p>	<p>The problem. We recognize it as being a circle in standard form.</p>
<p>$h = -5$ $k = 6$ $r^2 = 10$</p>	<p>You can read these variables straight out of the equation, just as in $y = mx + b$. Question: how can we make our equation's $(x + 5)$ look like the standard formula's $(x - h)$? Answer: if $h = -5$. In general, h comes out the opposite sign from the number in the equation. Similarly, $(y - 6)$ tells us that k will be positive 6.</p>
<p>Center: $(-5,6)$ Radius: $\sqrt{10}$</p>	<p>Now that we have the variables, we know everything we need to know about the circle.</p>
	<p>And we can graph it! $\sqrt{10}$ is, of</p>



course, just a little over 3—so we know where the circle begins and ends.

Just as you can go from a formula to a graph, you can also go the other way.

Example:
Find the Equation for this Circle

Find the equation for a circle with center at $(15,-4)$ and radius 8.

The problem.

$$(x - 15)^2 + (y + 4)^2 = 64$$

The solution, straight from the formula for a circle.

If a circle is given in **nonstandard form**, you can always recognize it by the following sign: it has both an x^2 and a y^2 term, and they have the **same coefficient**.

- $-3x^2 - 3y^2 + x - y = 5$ is a circle: the x^2 and y^2 terms both have the coefficient -3
- $3x^2 - 3y^2 + x - y = 5$ is **not** a circle: the x^2 term has coefficient 3, and the y^2 has -3
- $3x^2 + 3y = 5$ is **not** a circle: there is no y^2 term

Once you recognize it as a circle, you have to put it into the standard form for graphing. You do this by completing the square... twice!

**Example:
Graphing a Circle in Nonstandard Form**

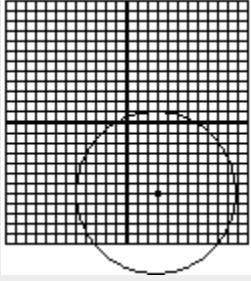
Graph $2x^2 + 2y^2 - 12x + 28y - 12 = 0$

The problem. The equation has both an x^2 and a y^2 term, and they have the **same coefficient** (a 2 in this case): this tells us it will graph as a circle.

$x^2 + y^2 - 6x + 14y - 6 = 0$

Divide by the coefficient (the 2). Completing the square is always easiest without a

	coefficient in front of the squared term.
$(x^2 - 6x) + (y^2 + 14y) = 6$	Collect the x terms together and the y terms together, with the number on the other side.
$(x^2 - 6x + 9) + (y^2 + 14y + 49) = 6 + 9 + 49$	Complete the square for both x and y .
$(x - 3)^2 + (y + 7)^2 = 64$	Rewrite our perfect squares. We are now in the correct form. We can see that this is a circle with center at $(3, -7)$ and radius 8. (*Remember How the signs change on h and k !)
	Once you have the center and radius, you can immediately



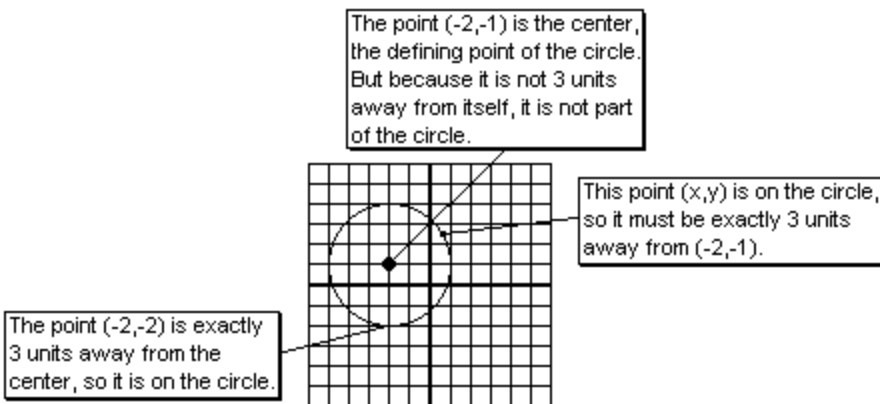
draw the circle, as we did in the previous example.

Going From the Definition of a Circle to the Formula

If you're following all this, you're now at the point where you understand the **definition** of a circle...and you understand the **formula** for a circle. But the two may seem entirely unconnected. In other words, when I said $(x-h)^2 + (y-k)^2 = r^2$ is the formula for a circle, you just had to take my word for it.

In fact, it is possible to **start** with the definition of a circle, and **work from there** to the formula, thus showing why the formula works the way it does.

Let's go through this exercise with a specific example. Suppose we want to find the formula for the circle with center at $(-2,1)$ and radius 3. We will start with the definition: this circle is the set of **all the points that are exactly 3 units away from the point $(-2,1)$** . Think of it as a club. If a point is exactly 3 units away from $(-2,1)$, it gets to join the club; if it is not exactly 3 units away, it doesn't get to join.



You already know what the formula is going to be, but remember, in this exercise we're not going to **assume** that formula—we're going to **assume** nothing but the definition, and **work our way** to the formula. So here is our starting point, the definition for this circle:

“The distance from (x, y) to $(-2, 1)$ is 3.”

Any point (x, y) that meets this criterion is in our club. Using the distance formula that we developed above, we can immediately translate this English language definition into a mathematical formula. Recall that if xxx_d is the distance between the points (x_1, y_1) and (x_2, y_1) , then $(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$ (Pythagorean Theorem). So in this particular case,

$$(x + 2)^2 + (y - 1)^2 = 9$$

Note that this corresponds perfectly to the formula given above. In fact, if you repeat this exercise more generically—using (h, k) as the center instead of $(-2, 1)$, and r as the radius instead of 3—then you end up with the exact formula given above, $(x - h)^2 + (y - k)^2 = r^2$.

For each of the remaining shapes, I'm going to repeat the pattern used here for the circle. First I will give the **geometric definition** and then the **mathematical formula**. However, I will not take the third step, of showing how the definition (with the distance formula) leads to the formula: you will do this, for each shape, in the exercises in the text.

Parabolas

This module discusses parabolas and how they relate to conic sections in Algebra. Includes example problems.

The Definition of a Parabola

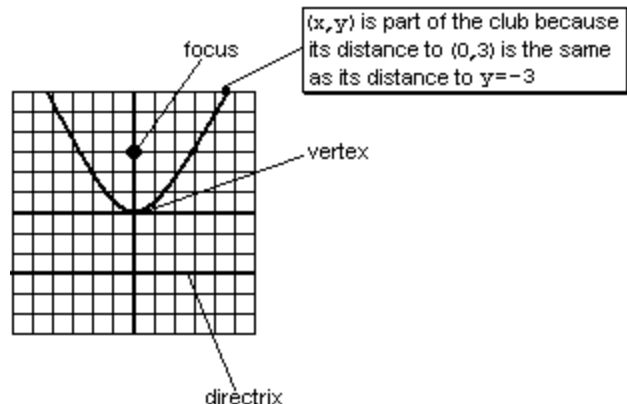
Based on the discussion of circles, you might guess that the definition of a parabola will take the form: “The set of all points that...” and you would be correct. But the definition of a parabola is more complicated than that of a circle.

Definition of a Parabola

Take a point (called the **focus**) and a horizontal line (the **directrix**) that does not contain that point. The set of all points in a plane that are the **same distance from the focus as from the directrix** forms a parabola.

In the text, you begin with a specific example of this process. The focus is $(0,3)$ and the directrix is the line $y = -3$. If we use our “club” analogy again, we could say that this time, a point is a member of our club if its distance to $(0,3)$ is the same as its distance to $y = -3$.

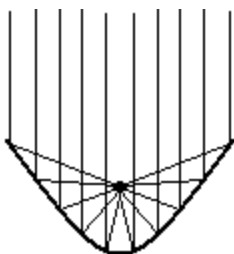
The resulting shape looks something like this:



You may recall that a circle is entirely defined by its center—but the center is not, itself, a part of the circle. In a similar way, the focus and directrix **define** a parabola; but neither the focus, nor any point on the directrix, is a

part of the parabola. The vertex, on the other hand—the point located directly between the focus and the directrix—is a part of the parabola.

One of the obvious questions you might ask at this point is—who cares? It's pretty obvious that circles come up a lot in the real world, but parabolas? It turns out that parabolas are more useful than you might think. For instance, many telescopes are based on parabolic mirrors. The reason is that all the light that comes in bounces off the mirror to the focus. The focus therefore becomes a point where you can see very dim, distant objects.



Parabolic mirror in a telescope.
All parallel light beams converge at the focus.

The Formula of a Parabola

We've already graphed parabolas in a previous chapter. As you may recall, we began with the simplest parabola, $y = x^2$, and **permuted** it.

- $x^2 + k$ moves it up by k
- $(x - h)^2$ moves it to the right by h
- Multiplying by a number in front stretches the graph vertically
- Multiplying by a **negative** number turns the graph upside-down.

Putting it all together, we arrive at:

Mathematical Formula for a Vertical Parabola

$y = a(x - h)^2 + k$ is a parabola with vertex (h, k) . If a is positive, it opens up; if a is negative, it opens down.

Parabolas can also be **horizontal**. For the most part, the concepts are the same. The simplest horizontal parabola is $x = y^2$, which has its vertex at

the origin and opens to the right—from there, you can permute it. The directrix in this case is a vertical line.

Mathematical Formula for a Horizontal Parabola

$x = a(y - k)^2 + h$ is a parabola with vertex (h, k) . If a is positive, it opens to the right; if a is negative, it opens to the left.

At this point, there are two useful exercises that you may want to try.

First, compare the two equations. How are they alike, and how are they different?

Second, consider the **horizontal** parabola equation as a set of permutations of the basic form $x = y^2$. What is k doing to the parabola, and why? How about h , and a ?

Ellipses

This module introduces the definition and formula of an ellipse, including example problems.

The Definition of an Ellipse

An ellipse is a sort of squashed circle, sometimes referred to as an oval.

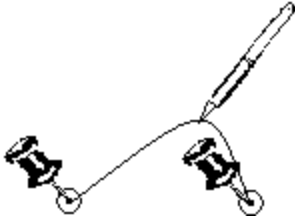
Definition of an Ellipse

Take two points. (Each one is a **focus**; together, they are the **foci**.) An ellipse is the set of all points in a plane that have the following property: the distance from the point to one focus, **plus** the distance from the point to the other focus, is some constant.

They just keep getting more obscure, don't they? Fortunately, there is an experiment you can do, similar to the circle experiment, to show why this definition leads to an elliptical shape.

Experiment: Drawing the Perfect Ellipse

1. Lay a piece of cardboard on the floor.
2. Thumbtack one end of a string to the cardboard.
3. Thumbtack the other end of the string, elsewhere on the cardboard.
The string should not be pulled taut: it should have some slack.
4. With your pen, pull the middle of the string back until it is taut.
5. Pull the pen all the way around the two thumbtacks, keeping the string taut at all times.
6. The pen will touch every point on the cardboard such that the distance to one thumbtack, plus the distance to the other thumbtack, is exactly one string length. And the resulting shape will be an ellipse. The cardboard is the "plane" in our definition, the thumbtacks are the "foci," and the string length is the "constant distance."



Do ellipses come up in real life? You'd be surprised how often. Here is my favorite example. For a long time, the orbits of the planets were assumed to be circles. However, this is incorrect: the orbit of a planet is actually in the shape of an ellipse. The sun is at one focus of the ellipse (**not** at the center). Similarly, the moon travels in an ellipse, with the Earth at one focus.

The Formula of an Ellipse

With ellipses, it is crucial to start by distinguishing **horizontal** from **vertical**.

Horizontal	Vertical
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Mathematical Formula for an Ellipse with its Center at the Origin

And of course, the usual rules of permutations apply. For instance, if we replace x with $x - h$, the ellipse moves to the right by h . So we have the more general form:

Horizontal	Vertical
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \ (a > b)$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \ (a > b)$

Mathematical Formula for an Ellipse with its Center at (h,k)

The key to understanding ellipses is understanding the three constants a , b , and c .

	Horizontal Ellipse	Vertical Ellipse
Where are the foci?	Horizontally around the center	Vertically around the center
How far are the foci from the center?	c	c
What is the “major axis”?	The long (horizontal) way across	The long (vertical) way across

	Horizontal Ellipse	Vertical Ellipse
How long is the major axis?	$2a$	$2a$
What is the “minor axis?”	The short (vertical) way across	The short (horizontal) way across
How long is the minor axis?	$2b$	$2b$
Which is biggest?	a is biggest. $a > b$, and $a > c$.	a is biggest. $a > b$, and $a > c$.
crucial relationship	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

The following example demonstrates how all of these concepts come together in graphing an ellipse.

**Example:
Graphing an Ellipse**

Graph $x^2 + 9y^2 - 4x + 54y + 49 = 0$

The problem. We recognize this as an ellipse because it has an x^2

	<p>and a y^2 term, and they both have the same sign (both positive in this case) but different coefficients (3 and 2 in this case).</p>
$x^2 - 4x + 9y^2 + 54y = -49$	<p>Group together the x terms and the y terms, with the number on the other side.</p>
$(x^2 - 4x) + 9(y^2 + 6y) = -49$	<p>Factor out the coefficients of the squared terms. In this case, there is no x^2 coefficient, so we just have to factor out the 9 from the y terms.</p>

$$(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -49 + 4 + 81$$

Complete the square twice. Remember, adding 9 **inside** those parentheses is equivalent to adding 81 to the left side of the equation, so we must add 81 to the right side of the equation!

$$(x - 2)^2 + 9(y + 3)^2 = 36$$

Rewrite and simplify. Note, however, that we are still not in the standard form for an ellipse!

$$\frac{(x-2)^2}{36} + \frac{(y+3)^2}{4} = 1$$

Divide by 36. This is because we need a 1 on the right, to be in our standard form!

Center: (2,-3)

We read the center from

	<p>the ellipse the same way as from a circle.</p>
<p>$a = 6$ $b = 2$</p>	<p>Since the denominators of the fractions are 36 and 4, a and b are 6 and 2. But which is which? The key is that, for ellipses, a is always greater than b. The larger number is a and the smaller is b.</p>
<p>Horizontal ellipse</p>	<p>Going back to the equation, we see that the a^2 (the larger denominator) was under the x, and the b^2 (the smaller) was under the y. This means our equation is a</p>

starts b above
the center
and ends b
below, so its
length is
4. The foci
are about $5\frac{1}{2}$
from the
center.

Hyperbolas

This module introduces the definition and formula of a hyperbola, including example problems.

The Definition of a Hyperbola

A hyperbola is the strangest-looking shape in this section. It looks sort of like two back-to-back parabolas. However, those shapes are **not** exactly parabolas, and the differences are very important.

Surprisingly, the definition and formula for a hyperbola are very similar to those of an ellipse.

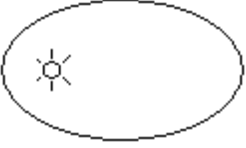
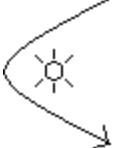
Definition of a Hyperbola

Take two points. (Each one is a **focus**; together, they are the **foci**.) A hyperbola is the set of all points in a plane that have the following property: the distance from the point to one focus, **minus** the distance from the point to the other focus, is some constant.

The entire definition is identical to the definition of an ellipse, with one critical change: the word “plus” has been changed to “minus.”

One use of hyperbolas comes directly from this definition. Suppose two people hear the same noise, but one hears it ten seconds earlier than the first one. This is roughly enough time for sound to travel 2 miles. So where did the sound originate? Somewhere **2 miles closer to the first observer than the second**. This places it somewhere on a hyperbola: the set of all points such that the distance to the second point, **minus** the distance to the first, is 2.

Another use is astronomical. Suppose a comet is zooming from outer space into our solar system, passing near (but not colliding with) the sun. What path will the comet make? The answer turns out to depend on the comet’s speed.

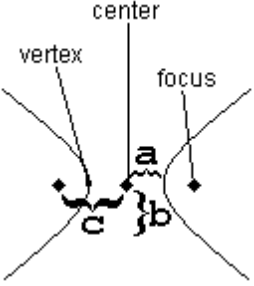
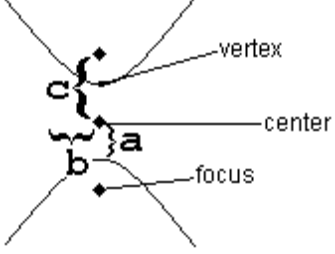
	
<p>If the comet's speed is low, it will be trapped by the sun's gravitational pull. The resulting shape will be an elliptical orbit.</p>	<p>If the comet's speed is high, it will escape the sun's gravitational pull. The resulting shape will be half a hyperbola.</p>

We see in this real life example, as in the definitions, a connection between ellipses and hyperbolas.

The Formula of an Hyperbola

With hyperbolas, just as with ellipses, it is crucial to start by distinguishing **horizontal** from **vertical**. It is also useful to pay close attention to which aspects are the same as ellipses, and which are different.

Horizontal	Vertical
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Horizontal	Vertical
 <p>A diagram of a horizontal hyperbola opening left and right. The center is at the origin. The vertices are on the x-axis, and the foci are further out on the x-axis. A right triangle is formed with vertices at the center, a vertex, and a focus. The horizontal leg is labeled 'a', the vertical leg is labeled 'b', and the hypotenuse is labeled 'c'. Labels 'center', 'vertex', and 'focus' point to their respective parts.</p>	 <p>A diagram of a vertical hyperbola opening up and down. The center is at the origin. The vertices are on the y-axis, and the foci are further out on the y-axis. A right triangle is formed with vertices at the center, a vertex, and a focus. The vertical leg is labeled 'a', the horizontal leg is labeled 'b', and the hypotenuse is labeled 'c'. Labels 'vertex', 'center', and 'focus' point to their respective parts.</p>

Mathematical Formula for a Hyperbola with its Center at the Origin

And of course, the usual rules of permutations apply. For instance, if we replace x with $x-h$, the hyperbola moves to the right by h . So we have the more general form:

Horizontal	Vertical
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Formula for a Hyperbola with its Center at xxx(h,k)

The key to understanding hyperbolas is understanding the three constants a , b , and c .

	Horizontal Hyperbola	Vertical Hyperbola
Where are the foci?	Horizontally around the center	Vertically around the center
How far are the foci from the center?	c	c
What is the “transverse axis”?	The (horizontal) line from one vertex to the other	The (vertical) line from one vertex to the other
How long is the transverse axis?	$2a$	$2a$
Which is biggest?	c is biggest. $c > a$, and $c > b$.	c is biggest. $c > a$, and $c > b$.
crucial relationship	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$

Having trouble keeping it all straight? Let’s make a list of similarities and differences.

Similarities between Hyperbolas and Ellipses

- The formula is identical, except for the replacement of a^+ with a^- .
- The definition of a is very similar. In a horizontal ellipse, you move horizontally a from the center to the edges of the ellipse. (This defines the major axis.) In a horizontal hyperbola, you move horizontally a from the center to the vertices of the hyperbola. (This defines the transverse axis.)
- b defines a different, perpendicular axis.
- The definition of c is identical: the distance from center to focus.

Differences Between Hyperbolas and Ellipses

- The biggest difference is that for an ellipse, a is always the biggest of the three variables; for a hyperbola, c is always the biggest. This should be evident from looking at the drawings (the foci are inside an ellipse, outside a hyperbola). However, this difference leads to several other key distinctions.
- For ellipses, $a^2 = b^2 + c^2$. For hyperbolas, $c^2 = a^2 + b^2$.
- For ellipses, you tell whether it is horizontal or vertical by looking at which denominator is greater, since a must always be bigger than b . For hyperbolas, you tell whether it is horizontal or vertical by looking at which variable has a positive sign, the x^2 or the y^2 . The relative sizes of a and b do not distinguish horizontal from vertical.

In the example below, note that the process of **getting the equation in standard form** is identical with hyperbolas and ellipses. The extra last step—rewriting a multiplication by 4 as a division by $\frac{1}{4}$ —can come up with ellipses as easily as with hyperbolas. However, it did not come up in the last example, so it is worth taking note of here.

Example:
Putting a Hyperbola in Standard Form

Graph $3x^2 - 12y^2 - 18x - 24y + 12 = 0$

The problem. We recognize this as a hyperbola because it has an x^2 and a y^2 term, and have

	<p>different signs (one is positive and one negative).</p>
$3x^2 - 18x - 12y^2 - 24y = -12$	<p>Group together the x terms and the y terms, with the number on the other side.</p>
$3(x^2 - 6x) - 12(y^2 + 2y) = -12$	<p>Factor out the coefficients of the squared terms. In the case of the y^2 for this particular equation, the coefficient is minus 12.</p>
$3(x^2 - 6x + 9) - 12(y^2 + 2y + 1) = -12 + 27 - 12$	<p>Complete the square twice. Adding 9 inside the first parentheses adds 27; adding 1</p>

	inside the second set subtracts 12.
$3(x - 3)^2 - 12(y + 1)^2 = 3$	Rewrite and simplify.
$(x - 3)^2 - 4(y + 1)^2 = 1$	Divide by 3, to get a 1 on the right. Note, however, that we are still not in standard form, because of the 4 that is multiplied by $(y + 1)^2$. The standard form has numbers in the denominator, but not in the numerator.
$(x - 3)^2 - \frac{(y+1)^2}{\frac{1}{4}} = 1$	Dividing by $\frac{1}{4}$ is the same as multiplying by 4, so this is still the same

equation.
But now we are in standard form, since the number is on the bottom.

However, the process of **graphing** a hyperbola is quite different from the process of graphing an ellipse. Even here, however, some similarities lurk beneath the surface.

**Example:
Graphing a Hyperbola in Standard Form**

Graph $(x - 3)^2 - \frac{(y+1)^2}{\frac{1}{4}} = 1$	The problem, carried over from the example above, now in standard form.
Center: (3,-1)	Comes straight out of the equation, both signs changed, just as with circles and ellipses.
$a = 1$ $b = \frac{1}{2}$	The square roots of the denominators, just as with the ellipse. But how do we tell

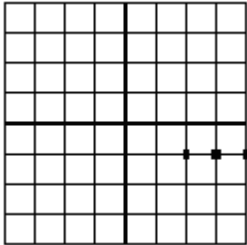
which is which? In the case of a hyperbola, the a always goes with the **positive term**. In this case, the x^2 term is positive, so the term under it is a^2 .

Horizontal hyperbola

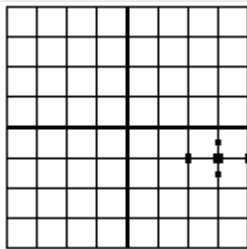
Again, this is because the x^2 term is positive. If the y^2 were the positive term, the hyperbola would be vertical, and the number under the y^2 term would be considered a^2 .

$$c = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

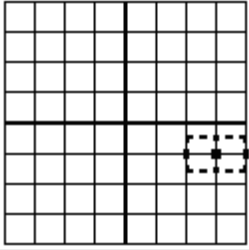
Remember that the relationship is different: for hyperbolas, $c^2 = a^2 + b^2$



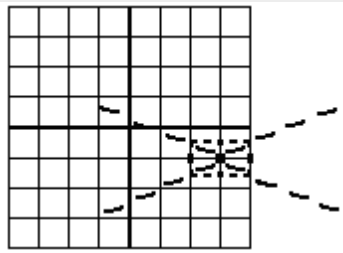
Now we begin drawing. Begin by drawing the center at (3, -1). Now, since this is a horizontal ellipse, the vertices will be aligned horizontally around the center. Since $a = 1$, move 1 to the left and 1 to the right, and draw the vertices there.



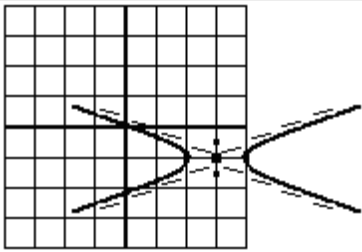
In the other direction—vertical, in this case—we have something called the “conjugate axis.” Move up and down by b ($\frac{1}{2}$ in this case) to draw the endpoints of the conjugate axis. Although not part of the hyperbola, they will help us draw it.



Draw a box that goes through the vertices and the endpoints of the conjugate axis. The box is drawn in dotted lines to show that it is not the hyperbola.



Draw diagonal lines through the corners of the box—also dotted, because they are also not the hyperbola. These lines are called the **asymptotes**, and they will guide you in drawing the hyperbola. **The further it gets from the vertices, the closer the hyperbola gets to the asymptotes. However, it never crosses them.**



Now, at last, we are ready to draw the hyperbola. Beginning at the vertices, approach—but do not cross!—the asymptotes. So you see that the asymptotes guide us in setting the **width** of the hyperbola, performing a similar function to the latus rectum in parabolas.

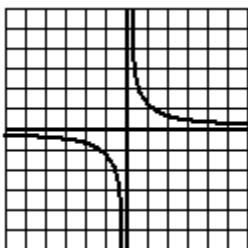
The hyperbola is the most complicated shape we deal with in this course, with a lot of steps to memorize.

But there is also a very important concept hidden in all that, and that is the concept of an **asymptote**. Many functions have asymptotes, which you will

explore in far greater depth in more advanced courses. An asymptote is a line that a function approaches, but never quite reaches. The asymptotes are the easiest way to confirm that a hyperbola is **not** actually two back-to-back parabolas. Although one side of a hyperbola resembles a parabola superficially, parabolas do not have asymptotic behavior—the shape is different.

Remember our comet? It flew into the solar system at a high speed, whipped around the sun, and flew away in a hyperbolic orbit. As the comet gets farther away, the sun’s influence becomes less important, and the comet gets closer to its “natural” path—a straight line. In fact, that straight line is the asymptote of the hyperbolic path.

Before we leave hyperbolas, I want to briefly mention a much simpler equation: $y = \frac{1}{x}$. This is the equation of a **diagonal** hyperbola. The asymptotes are the x and y axes.



$$y = \frac{1}{x}$$

Although the equation looks completely different, the shape is identical to the hyperbolas we have been studying, except that it is rotated 45° .

Conic Concepts -- Recap

If it has...	Then it's a...	Example	Horizontal or Vertical?
No squared terms	Line	$2x + 3y = 7$	
One squared term	Parabola	$2x^2 - 10x + 7y = 9$	If you have an x^2 but no y^2 , you're a horizontal parabola. If you have a y^2 but no x^2 , vertical.
Two squared terms with the same coefficient	Circle	$3x^2 + 3y^2 + 6x + 3y = 2$	

If it has...	Then it's a...	Example	Horizontal or Vertical?
Two squared terms with different coefficients but the same sign	Ellipse	$2x^2 + 3y^2 + 6x + 6y = 12$	The difference between vertical ellipses and horizontal is based on which squared term has the larger coefficient.
Two squared terms with different signs	Hyperbola	$3x^2 - 3y^2 + 6x + 3y = 2$	The difference between vertical hyperbolas and horizontal is based on which squared term is positive.

Note that all of this is based **only on the squared terms!** The other terms matter in terms of graphing, but not in terms of figuring out what shape it is.

Sequences

An updated version of the Sequences module.

A **sequence** is a list of numbers: like 4,9,3,2,17.

An **arithmetic sequence** is a list where each number is generated by adding a constant to the previous number. An example is 10,13,16,19,22,25. In this example, the first term (t_1) is 10, and the “common difference” (d)—that is, the difference between any two adjacent numbers—is 3. Another example is 25,22,19,16,13,10. In this example $t_1 = 25$, and $d = (-3)$. In both of these examples, n (the number of terms) is 6.

A **geometric sequence** is a list where each number is generated by multiplying a constant by the previous number. An example is 2,6,18,54,162. In this example, $t_1 = 2$, and the “common ratio” (r)—that is, the ratio between any two adjacent numbers—is 3. Another example is 162,54,18,6,2. In this example $t_1 = 162$, and $r = \frac{1}{3}$. In both examples $n = 5$.

A **recursive definition** of a sequence means that you define each term based on the previous. So the recursive definition of an arithmetic sequence is $t_n = t_{n-1} + d$, and the recursive definition of a geometric sequence is $t_n = rt_{n-1}$.

An **explicit definition** of an arithmetic sequence means you define the n^{th} term without making reference to the previous term. This is more useful, because it means you can find (for instance) the 20th term without finding all the other terms in between.

To find the explicit definition of an arithmetic sequence, you just start writing out the terms. The first term is always t_1 . The second term goes up by d so it is $t_1 + d$. The third term goes up by d again, so it is $(t_1 + d) + d$, or in other words, $t_1 + 2d$. So we get a chart like this.

t_1	t_2	t_3	t_4	t_5
t_1	$t_1 + d$	$t_1 + 2d$	$t_1 + 3d$	$t_1 + 4d$

...and so on. From this you can see the generalization that $t_n = t_1 + (n - 1)d$, which is the explicit definition we were looking for.

The explicit definition of a geometric sequence is arrived at the same way. The first term is t_1 ; the second term is r times that, or t_1r ; the third term is r times **that**, or t_1r^2 ; and so on. So the general rule is $t_n = t_1r^{n-1}$. Read this as: “ t_1 multiplied by r , $(n-1)$ times.”

Series

An updated version of the Series module.

A **series** is a list of numbers—like a sequence—but instead of listing them, you add them all up. For instance, $4+9+3+2+17$. (This particular series adds up to 35.)

One way to compactly represent a series is with “summation notation,” which looks like this:

Equation:

$$\sum_{n=3}^7 n^2$$

The big funny-looking thing in the middle is the Greek letter uppercase Sigma, and it indicates a series. To “unpack” this notation, start counting at the bottom ($n = 3$), and stop when you reach the stop ($n = 7$). For each term, plug that value of n into the given formula (n^2). So this particular formula, which we can read as “the sum as n goes from 3 to 7 of n^2 ”, simply means:

$$3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

Arithmetic Series

If you add up all the terms of an arithmetic sequence, you have an arithmetic series. For instance, $10 + 13 + 16 + 19 + 22 + 25 = 105$.

There is a “trick” that can be used to add up the terms of any arithmetic series. While this trick may not save much time with a 6-item series like the one above, it can be very useful if adding up longer series. The trick is to work **from the outside in**.

Consider the example given above: $10 + 13 + 16 + 19 + 22 + 25$. Looking at the first and last terms, $10 + 25 = 35$. Going in, to the second

and next-to-last terms, $13 + 22 = 35$. Finally, the two inside numbers $16 + 19 = 35$. So we can see that the sum of the whole thing is $3 \cdot 35$.

Pause here and check the following things.

- You understand the calculation that was done for this particular example.
- You understand that this “trick” will work for any arithmetic series.
- You understand that this trick will **not** work, in general, for series that are not arithmetic.

If we apply this trick to the **generic** arithmetic series, we get a formula that can be used to sum up any arithmetic series.

Every arithmetic series can be written as follows:

$$t_1 + (t_1 + d) + (t_1 + 2d) \dots (t_n - d) + t_n$$

If you add the first and last terms, you get $t_1 + t_n$. Ditto for the second and next-to-last terms, and so on. How many such pairs will there be in the whole series? Well, there are n terms, so there are $\frac{n}{2}$ pairs. So the sum for the whole series is $\frac{n}{2} (t_1 + t_n)$.

Geometric Series

If you add up all the terms of a geometric sequence, you have a geometric series. The “arithmetic series trick” will not work on such a series; however, there is a **different** trick we can use. As an example, let’s find the sum $2 + 6 + 18 + 54 + 162$.

We begin by calling the sum of this series S :

$$S = 2 + 6 + 18 + 54 + 162$$

Now, if you multiply both sides of this equation by 3, you get the first equation I have written below. (The second equation below is just copied from above.)

$$3S = 6 + 18 + 54 + 162 + 486 \text{ (*confirm this for yourself!)}$$

$$S = 2 + 6 + 18 + 54 + 162$$

Here comes the key moment in the trick: **subtract the two equations**. This leaves you with:

$$2S = 486 - 2, \text{ so } S = 242$$

Once again, pause to convince yourself that this will work on all geometric series, but only on geometric series.

Finally—once again—we can apply this trick to the **generic** geometric series to find a formula. So we begin with $t_1 + t_1r + t_1r^2 + t_1r^3 \dots t_1r^{n-1}$ and write...

$$rS = t_1r + t_1r^2 + t_1r^3 + t_1r^4 \dots t_1r^{n-1} + t_1r^n \text{ (*confirm this!)}$$

$$S = t_1 + t_1r + t_1r^2 + t_1r^3 + \dots t_1r^{n-1}$$

Again, subtracting and solving, we get...

$$rS - S = t_1r^n - t_1$$

$$S(r - 1) = t_1(r^n - 1)$$

$$S = t_1 \frac{r^n - 1}{r - 1}$$

So there we have it: a general formula for the sum of any finite geometric series, with the first term t_1 , the common ratio r , and a total of n terms.

Proof by Induction

An updated version of the Proof by Induction module.

“Induction” is a method of proving something. Once again, let’s start with an example.

Consider the sum $\sum_{i=1}^n \frac{1}{i(i+1)}$. In other words,
 $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)}$. This is neither arithmetic nor geometric, so none of our established tricks will work on it. How can we find the sum of such a series?

Students are often surprised to hear that mathematicians typically begin such problems by **looking for a pattern**. What does this series do for the first few terms?

- 1 term: $\frac{1}{1 \times 2} = \frac{1}{2}$
- 2 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
- 3 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

At this point, you might already suspect the pattern. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$...could it be that the next term will be $\frac{4}{5}$? Let’s find out.

- 4 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$

It seems to work. The next term will probably be $\frac{5}{6}$, and then $\frac{6}{7}$, and so on. Stop for a moment and make sure you see the pattern. Then, see if you can express that pattern using mathematical notation instead of words. (Try this yourself before you keep reading!)

The pattern can be expressed like this:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Stop for a moment and make sure you know where we are. What we have done is figured out a **pattern** to the answers, and shown that the pattern works for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. Based on this pattern, we

expect that if we added up $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{100 \times 101}$, we would get $\frac{100}{101}$.

But we have not yet proven anything. Maybe the pattern breaks down for $n = 5$. Or maybe it works for all the n -values from 1 to 1000, and then suddenly stops working. We cannot possibly test all the values in the world, one by one.

This is where the proof by induction comes in. It gives us a way to prove that such a pattern will continue to hold forever.

An inductive proof, in general, consists of two steps. The first step is to show that the pattern holds when $n = 1$. The second step is to show that, **whenever this pattern holds for some particular n , it will also hold for the next n .** If it holds for $n = 5$, then it must hold for $n = 6$. If it works for $n = 99$, then it must also work for $n = 100$. Once we have proven that, in general, then we will have shown that it works for **all** n values.

Example:

Proof by Induction

Inductive Proof of:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

First Step:

Show that it works for $n = 1$

$$1 \text{ term: } \frac{1}{1 \times 2} = \frac{1}{2}$$



Second Step:

Show that it works for $n + 1$, assuming it works for some n

For $n + 1$, the left side of the equation looks like:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+1+1)}$$

and the right side of the equation looks like:

Equation:

$$\frac{n + 1}{n + 1 + 1}$$

So we want to see if:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}$$

?

$$\frac{n+1}{n+2}$$

Now comes the key step.

If this pattern held for n , then the first n terms of the left side—that is, all but the last (new) term—must add up to $\frac{n}{n+1}$. So we do that substitution:

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

?

$$\frac{n+1}{n+2}$$

All that remains, now, is the algebra to show that equation is true. Get a common denominator:

$$\frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

?

$$\frac{(n+1)^2}{(n+1)(n+2)}$$

$$n^2 + 2n + 1$$

✓

$$(n + 1)^2$$

✓

The algebra here all comes from our unit on rational expressions: you may want to take a moment to make sure you can follow it. But don't let the algebra distract you from the main point, which is what we proved in the

second step. We proved that the formula works for $n + 1$. But in the middle of that proof, we **assumed** that it works for the previous term, n . In doing so, we proved that if it works for 1, it must also work for 2; if it works for 2, it must also work for 3; and so on. This amounts, then, to a proof that the pattern holds forever.

Probability

An introduction to probability and the multiplication rule.

If you flip a coin, what is the chance of getting heads? That's easy: 50/50. In the language of probability, we say that the probability is $\frac{1}{2}$. That is to say, half the time you flip coins, you will get heads.

So here is a harder question: if you flip **two** coins, what is the chance that you will get heads **both times**? I asked this question of my son, who has good mathematical intuition but no training in probability. His immediate answer: $\frac{1}{3}$. There are three possibilities: two heads, one heads and one tails, and two tails. So there is a $\frac{1}{3}$ chance of getting each possibility, including two heads. Makes sense, right?

But it is not right. If you try this experiment 100 times, you will not find about 33 "both heads" results, 33 "both tails," and 33 "one heads and one tails." Instead, you will find something much closer to: 25 "both heads," 25 "both tails," and 50 "one of each" results. Why?

Because hidden inside this experiment are actually **four different results**, each as likely as the others. These results are: heads-heads, heads-tails, tails-heads, and tails-tails. Even if you don't keep track of what "order" the coins flipped in, heads-tails is still a different result from tails-heads, and each must be counted.

And what if you flip a coin **three** times? In this case, there are actually eight results. In case this is getting hard to keep track of, here is a systematic way of listing all eight results.

First Coin	Second Coin	Third Coin	End Result
Heads	Heads	Heads	HHH

First Coin	Second Coin	Third Coin	End Result
	Tails	Tails	HHT
		Heads	HTH
		Tails	HTT
Tails	Heads	Heads	THH
		Tails	THT
	Tails	Heads	TTH
		Tails	TTT

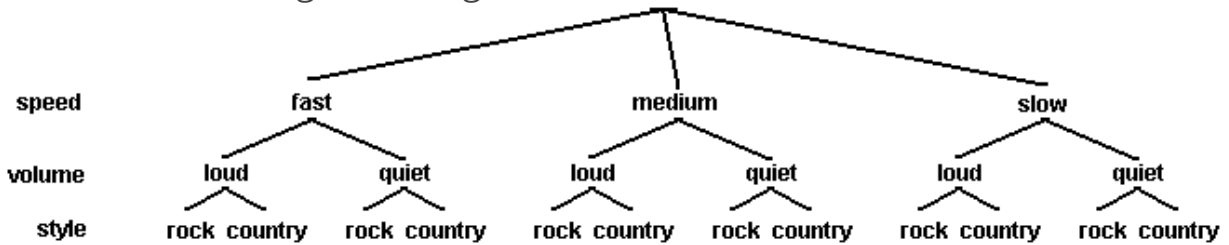
When you make a table like this, the pattern becomes apparent: each new coin doubles the number of possibilities. The chance of three heads in a row is $\frac{1}{8}$. What would be the chance of four heads in row?

Let’s take a slightly more complicated—and more interesting—example. You are the proud inventor of the SongWriter 2000tm.



The user sets the song speed (“fast,” “medium,” or “slow”); the volume (“loud” or “quiet”); and the style (“rock” or “country”). Then, the SongWriter automatically writes a song to match.

How many possible settings are there? You might suspect that the answer is $3 + 2 + 2 = 7$, but in fact there are many more than that. We can see them all on the following “tree diagram.”



If you start at the top of a tree like this and follow all the way down, you end up with one particular kind of song: for instance, “fast loud country song.” There are 12 different song types in all. This comes from **multiplying** the number of settings for each knob: $3 \times 2 \times 2 = 12$.

Now, suppose the machine has a “Randomize” setting that randomly chooses the speed, volume, and style. What is the probability that you will end up with a loud rock song that is **not** slow? To answer a question like this, you can use the following process.

1. Count the total number of results (the “leaves” in the tree) that match your criterion. In this case there are 2: the “fast-loud-rock” and “medium-loud-rock” paths.
2. Count the total number of results: as we said previously, there are 12.
3. Divide. The probability of a non-slow loud rock song is $2/12$, or $1/6$.

Note that this process will always give you a number between 0 (no results match) and 1 (all results match). Probabilities are always between 0 (for something that never happens) and 1 (for something that is guaranteed to happen).

But what does it really mean to say that “the probability is $1/6$?” You aren’t going to get $1/6$ of a song. One way to make this result more concrete is to imagine that you run the machine on its “Randomize” setting 100 times. You should expect to get non-slow loud rock songs 1 out of every 6 times; roughly 17 songs will match that description. This gives us another way to express the answer: there is a 17% probability of any given song matching this description.

The multiplication rule

We can look at the above problem another way.

What is the chance that any given, randomly selected song will be non-slow? $\frac{2}{3}$. That is to say, 2 out of every three randomly chosen songs will be non-slow.

Now...**out of those** $\frac{2}{3}$, how many will be loud? Half of them. The probability that a randomly selected song is both non-slow and loud is half of $\frac{2}{3}$, or $\frac{1}{2} \times \frac{2}{3}$, or $\frac{1}{3}$.

And now, **out of that** $\frac{1}{3}$, how many will be rock? Again, half of them: $\frac{1}{2} \times \frac{1}{3}$. This leads us back to the conclusion we came to earlier: 1/6 of randomly chosen songs will be non-slow, loud, rock songs. But it also gives us an example of a very general principle that is at the heart of all probability calculations:

When two events are independent, the probability that they will both occur is the probability of one, multiplied by the probability of the other.

What does it mean to describe two events as “independent?” It means that they have no effect on each other. In real life, we know that rock songs are more likely to be fast and loud than slow and quiet. Our machine, however, keeps all three categories independent: choosing “Rock” does not make a song more likely to be fast or slow, loud or quiet.

In some cases, applying the multiplication rule is very straightforward. Suppose you generate two different songs: what is the chance that they will **both** be fast songs? The two songs are independent of each other, so the chance is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

Now, suppose you generate five different songs. What is the chance that they will **all** be fast? $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$, or $(\frac{1}{3})^5$, or 1 chance in 243. Not very likely, as you might suspect!

Other cases are less obvious. Suppose you generate five different songs. What is the probability that **none** of them will be a fast song? The multiplication rule tells us only how to find the probability of “this **and** that”; how can we apply it to this question?

The key is to reword the question, as follows. What is the chance that the first song will not be fast, **and** the second song will not be fast, **and** the third song will not be fast, and so on? Expressed in this way, the question is a perfect candidate for the multiplication rule. The probability of the first song being non-fast is $\frac{2}{3}$. Same for the second, and so on. So the probability is $(\frac{2}{3})^5$, or 32/243, or roughly 13%.

Based on this, we can easily answer another question: if you generate five different songs, what is the probability that **at least one** of them will be fast? Once again, the multiplication rule does not apply directly here: it tells us “this and that,” not “this or that.” But we can recognize that this is the opposite of the previous question. We said that 13% of the time, none of the songs will be fast. That means that the **other** 87% of the time, at least one of them will!

Permutations

An introduction to permutations.

In the game of “Solitaire” (also known as “Patience” or “Klondike”), seven cards are dealt out at the beginning, as shown to the left: one face-up, and the other six face-down. (A bunch of other cards are dealt out too, but let’s ignore that right now.) A complete card deck has 52 cards. Assuming that all you know is the 7 of spades showing, how many possible “hands” (the other six cards) could be showing underneath? What makes this a “permutations” problem is that **order matters**: if an ace is hiding somewhere in those six cards, it makes a big difference if the ace is on the first position, the second, **etc.** Permutations problems can always be addressed as an example of the multiplication rule, with one small twist.

- **Question** How many cards might be in the first position, directly under the showing 7?
- **Answer** 51. That card could be anything except the 7 of spades.
- **Question** For any given card in first position, how many cards might be in second position?
- **Answer** 50. The seven of spades, and the next card, are both “spoken for.” So there are 50 possibilities left in this position.
- **Question** So how many possibilities are there for the first two positions combined?
- **Answer** 51×50 , or 2,550.
- **Question** So how many possibilities are there for all six positions?
- **Answer** $51 \times 50 \times 49 \times 48 \times 47 \times 46$, or approximately 1.3×10^{10} ; about 10 **billion** possibilities!

This result can be expressed (and typed into a calculator) more concisely by using **factorials**.

A “factorial” (written with an exclamation mark) means “multiply all the numbers from 1 up to this number.” So $5!$ means $1 \times 2 \times 3 \times 4 \times 5 = 120$

.

What is $\frac{7!}{5!}$? Well, it is $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5}$, of course. Most of the terms cancel, leaving only $6 \times 7 = 42$.

And what about $\frac{51!}{45!}$? If you write out all the terms, you can see that the first 45 terms cancel, leaving $46 \times 47 \times 48 \times 49 \times 50 \times 51$, which is the number of permutations we want. So instead of typing into your calculator six numbers to multiply (or sixty numbers or six hundred, depending on the problem), you can always find the answer to a permutation problem by dividing two factorials. In many calculators, the factorial option is located under the “probability” menu for this reason.

Combinations

An introduction to combinations.

Let's start once again with a deck of 52 cards. But this time, let's deal out a poker hand (5 cards). How many possible poker hands are there?

At first glance, this seems like a minor variation on the Solitaire question above. The only real difference is that there are five cards instead of six. But in fact, there is a more important difference: **order does not matter**. We do not want to count "Ace-King-Queen-Jack-Ten of spades" and "Ten-Jack-Queen-King-Ace of spades" separately; they are the same poker hand.

To approach such question, we **begin** with the permutations question: how many possible poker hands are there, if order **does** matter?

$52 \times 51 \times 50 \times 49 \times 48$, or $\frac{52!}{47!}$. But we know that we are counting every possible hand many different times in this calculation. How many times?

The key insight is that this second question—"How many different times are we counting, for instance, Ace-King-Queen-Jack-Ten of spades?"—is itself a permutations question! It is the same as the question "How many different ways can these five cards be rearranged in a hand?" There are five possibilities for the first card; for each of these, four for the second; and so on. The answer is $5!$ which is 120. So, since we have counted every possible hand 120 times, we divide our earlier result by 120 to find that there are $\frac{52!}{(47!)(5!)}$, or about 2.6 Million possibilities.

This question—"how many different 5-card hands can be made from 52 cards?"—turns out to have a surprisingly large number of applications. Consider the following questions:

- A school offers 50 classes. Each student must choose 6 of them to fill out a schedule. How many possible schedules can be made?
- A basketball team has 12 players, but only 5 will start. How many possible starting teams can they field?
- Your computer contains 300 videos, but you can only fit 10 of them on your iPod. How many possible ways can you load your iPod?

Each of these is a **combinations** question, and can be answered exactly like our card scenario. Because this type of question comes up in so many different contexts, it is given a special name and symbol. The last question would be referred to as “300 choose 10” and written $\binom{300}{10}$. It is calculated, of course, as $\frac{300!}{(290!)(10!)}$ for reasons explained above.